Separable deformations of group algebras: The Donald–Flanigan Conjecture

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Blankenberge
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Can a group algebra kG always be deformed to a separable algebra?

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Does there exist a $k[[t]]$-algebra $[kG]_t$ with $[kG]_t/\langle t\rangle \cong kG$, such that the scalar extension $(kG)_t := k((t)) \otimes_{k[[t]]} [kG]_t$ is $k((t))$-separable?

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The [DF] Conjecture

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One can show that

$$(kC_n)_t := k((t))[x]/\langle x^n - tx - 1\rangle$$

is a separable deformation of kC_n.

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Given separable deformations $(kG_1)_t$ and $(kG_2)_t$ of some group algebras kG_1 and kG_2 respectively, the algebra $(kG_1)_t \otimes_{k((t))} (kG_2)_t$ is a separable deformation of $k(G_1 \times G_2)$.

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- When G has a normal Abelian p-Sylow subgroup and k as above (M. Gerstenhaber and M. Schaps (1996)).
- When G is a dihedral group (K. Erdmann and M. Schaps (1993)) or a semidihedral group (K. Erdmann (1994)).
- When G is a reflection group (with six exceptions) (M. Gerstenhaber, A. Giaquinto and M. Schaps (2001), M. Gerstenhaber and M. Schaps (1997), M. Peretz and M. Schaps (1999)).
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Note that even for the case of a semidirect product of cyclic groups $C_m \rtimes_\xi C_n$, we do not have an answer.

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Can we use this information in order to find a separable deformation of kG?

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This is a special case of a *crossed product*, and we denote

$$kN * C_p := kN[y;\eta]/\langle y^p - u\rangle.$$

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Suppose that (a) and (b) above are satisfied.
Then, there is a polynomial $q_t(y) \in t[kN]_t[y;\eta_t]$, such that

$$(kN)_t[y;\eta_t]/\langle y^p - u_t + q_t(y)\rangle$$

is a separable deformation of $kN * C_p$.

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For any odd $s > 3$ (with some exceptions), we thus obtain a new example of a group satisfying the [DF] Conjecture.

Thank you!