Beams and Scaffolds - The art of building modular Garside groups

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Groups, Rings and the Yang-Baxter Equation

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Presentation Overview

Introduction

Notation

Let $(L, \leq)$ be a poset and $A \subseteq L$. For $z \in L$, write

- $z = \bigwedge A$, if for all $y \in L$, we have the equivalence

$$(\forall x \in L : y \leq x) \Leftrightarrow y \leq z.$$

  We call z the meet of A.

- $z = \bigvee A$, if for all $y \in L$, we have the equivalence

$$(\forall x \in L : y \geq x) \Leftrightarrow y \geq z.$$

  We call z the join of A.

Definition: (Bounded) Lattice

Let L be a poset.

- L is a lattice if for all $x, y \in L$, the elements $x \vee y = \bigvee\{x, y\}$ and $x \wedge y = \bigwedge\{x, y\}$ exist.
- L bounded from above (resp. below) if $1_L = \bigwedge \emptyset$ (resp. $0_L = \bigvee \emptyset$) exist.
- L is bounded, if L is bounded from above and below.

Definition: Right ℓ-group

A right-ordered group is a group G with a partial order $\leq$ such that for all $x, y, z \in G$,

$$x \leq y \Rightarrow xz \leq yz.$$

If $(G, \leq)$ is a lattice, G is a right ℓ-group.
The negative cone of G is $G^{-} = \{x \in G : x \leq e\}$.

Definition: Strong order unit

Let G be a right ℓ-group. An element $s \in G$ is a strong order unit, if

- $x \leq y \Leftrightarrow sx \leq sy$ holds for all $x, y \in G$,
- for each $g \in G$ there is a $k \in \mathbb{Z}$ such that $s^k \geq g$.

The interval $[s^{-1}, e] = \{x \in G : s \leq x \leq e\}$ is the respective strong order interval.

Definition: Noetherian

A right ℓ-group G is noetherian if every sequence $x_1 \leq x_2 \leq \ldots$, all $x_i \leq e$, becomes stationary and every sequence $y_1 \geq y_2 \geq \ldots$, all $y_i \geq e$, becomes stationary.

Definition: (Quasi-)Garside group

- A quasi-Garside group is a noetherian right ℓ-group with a strong order unit s.
- A Garside group is a quasi-Garside group with a *finite* strong order interval $[s^{-1}, e]$.

Examples of Garside groups: Spherical Artin-Tits groups, $\mathbb{Z}^n$, ...

Some bad problems

- Classify all Garside groups! ☹
- Find all Garside structures on a given torsion-free group! ☹

Some better problems

- Find all Garside groups with a given lattice structure! ☺
- Classify all Garside groups whose lattices fulfill certain identities! ☺

A Garside-theorist's favorite lattice identities!

- Distributivity: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- Modularity: $x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$

- Distributive lattices: $(\mathcal{P}(X), \subseteq)$, $(\mathbb{Z}^n, \leq)$, ...
- Modular lattices: $L(R, n) = \{R\text{-submodules of } R^n\}$, R unital, under $\subseteq$, ...
- Distributivity $\Rightarrow$ Modularity.

Cycle sets

Definition: Cycle set

A nondegenerate cycle set $(X, \cdot)$ is a (finite) set with a binary operation $(x, y) \mapsto x \cdot y$, such that

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \quad \forall x, y, z \in X$$
$$\text{the maps } \sigma_x(y) = x \cdot y \text{ are bijective for all } x \in X$$
$$\text{the map } x \mapsto x \cdot x \text{ is bijective.}$$

The structure group of a cycle set X is

$$G(X) = \langle\, X \mid (x \cdot y)x = (y \cdot x)y \rangle\,.$$

- Non-degenerate cycle sets are equivalent to set-theoretic solutions to the Yang-Baxter equation,
- $G(X)$ is equivalent to the structure group of a set-theoretic solution.

The Chouraqui-Rump theorem

Theorem [Chouraqui (2011), Rump (2015)]

- Let X be a nondegenerate cycle set. Then the submonoid $G^- = \langle X^{-1} \rangle \subseteq G(X)$ is the negative cone of a Garside structure on $G(X)$, that has $s = \bigvee X$ as its strong order unit.
- Every distributive Garside group G is of this form.
- There is a lattice isomorphism $G \cong \mathbb{Z}^n$.

Modular quasi-Garside groups

An example

Example: Paraunitary groups

Let $A \in K^{n\times n}$ be such that $v \neq 0 \Rightarrow v^\top Av \neq 0$ for $v \in K^n$ and $A^\top = A$. Then the pure paraunitary group associated with A is

$$\mathrm{PPU}(A) = \left\{M(t) \in K[t,t^{-1}]^{n\times n} \;:\; M(t^{-1})^\top AM(t) = A \wedge M(1) = E_n\right\}.$$

Theorem [D. (2019)]

The submonoid $\mathrm{PPU}(A)^- = \mathrm{PPU}(A) \cap K[t^{-1}]^{n\times n}$ is the negative cone of a quasi-Garside structure on $\mathrm{PPU}(A)$.
The element $s = t \cdot E_n$ is a strong order unit and

$$[s^{-1}, e] \cong L(K, n).$$

- For a cyclic field extensions L/K of degree n, one can construct a quasi-Garside group with a strong order unit s such that $[s^{-1}, e] \cong L(K, n)$. This construction uses skew polynomial rings.
- Rings seem to play a role in the construction of modular quasi-Garside groups. . .

The distributive scaffold

For $x > y$, write $x \succ y$ if $x \geq z \geq y$ implies $z = x$ or $z = y$.

- Let G be a modular quasi-Garside group, where $s = \bigvee\{x \in G : x \succ e\}$ exists.
- There is a finite decomposition $[s^{-1}, e] \cong \prod_{i=1}^{k} L_i$ into directly irreducible bounded lattices.
- For $1 \leq i \leq k$, let $z_i \in [s^{-1}, e]$ correspond to $\varepsilon^{(i)} = (\varepsilon_j^{(i)})_{1 \leq j \leq k}$ where

$$\varepsilon_j^{(i)} = \begin{cases} 0_j & j = i \\ 1_j & j \neq i \end{cases}$$

and set $\mathcal{Z} = \{z_i : 1 \leq i \leq k\}$.

Theorem [D. (2023)]

$\mathcal{Z}$ has the structure a nondegenerate cycle set such that the (well-defined) group homomorphism $G(\mathcal{Z}) \to G$ is an embedding of $\mathcal{D}(G) = G(\mathcal{Z})$ as a distributive sublattice.

Definition: Distributive scaffold

The subgroup $\langle \mathcal{Z} \rangle$ is the distributive scaffold of G.

The decomposition theorem - What is a beam?

Theorem [D. (2023)]

Let G be a modular quasi-Garside group with strong order unit $s = \bigvee\{x \in G : x \succ e\}$. Let

$$[s^{-1}, e] \cong \prod_{i=1}^{k} L_i$$

be a decomposition of $[s^{-1}, e]$ into directly irreducible lattices. Then there exist directly irreducible sublattices $\beth_i \subseteq G$ $(1 \leq i \leq k)$ - the beams - such that there is a *lattice-theoretic* decomposition

$$G \cong \prod_{i=1}^{k} \beth_i$$

that induces the decompositions

- $[s^{-1}, e] \cong \prod_{i=1}^{k} L_i$
- $\mathcal{D}(G) \cong \prod_{i=1}^{k} \mathbb{Z}$.

$\Rightarrow$ The decomposition of modular quasi-Garside groups is controlled by $\mathcal{D}(G)$, the structure group of a cycle set!

Some facts

Let G be a modular quasi-Garside group with strong order unit $s = \bigvee\{x \in X : x \succ e\}$. Let $[s^{-1}, e] \cong \prod_{i=1}^{k} L_i$ be a decomposition into irreducible lattices.

- By a theorem of Rump, the L_i are bounded modular geometric lattices - i.e. isomorphic to $L(K, n)$ for some skew field K or the subspace lattice of a degenerate geometry or a non-desarguesian plane.
- The associated beams $\sqsupset_i$ can be shown to be primary lattices.

Theorem [D. (2023)]

Let $L_i \cong L(K, n)$ for some $n \geq 4$ and a skew field K. For the associated beam $\sqsupset_i$, there is a noncommutative discrete valuation field Q with valuation ring R, such that there is a lattice-isomorphism

$$\begin{aligned} \sqsupset_i &\cong \mathrm{Lat}(Q, n) \\ &= \left\{A \subseteq Q^n : A \text{ is a finitely generated, essential } R\text{-submodule}\right\}. \end{aligned}$$

Some consequences

Isotypical components

Let $G \cong \prod_{i=1}^{k} \beth_i$ be the decomposition into beams. Let $i \sim j \Leftrightarrow \beth_i \cong \beth_j$.
For $1 \leq i \leq k$, call $\mathcal{C}_i = \prod_{j \sim i} \beth_j$ the isotypical component of $\beth_i$.

Theorem [D. (2023)]

- $\mathcal{C}_i$ corresponds to a convex subgroup $G_i \leq G$.
- If $\beth_i \cong \mathrm{Lat}(Q, n)$, the isotypical subgroup G_i embeds as a subgroup of $\mathcal{S}_m \wr \mathrm{P\Gamma}L({}_RQ^n)$, m the number of $\beth_j \cong \beth_i$, where

$$\mathrm{P\Gamma}L({}_RQ^n) = (\mathrm{GL}(n, Q) \rtimes \mathrm{Aut}(R)) \,/ R^{\times}\,,$$

  a generalized projective semilinear group. More precisely,

$$\mathrm{P\Gamma}L({}_RQ^n) = G \cdot \mathrm{P\Gamma}L({}_RR^n) \quad ; \quad G \cap \mathrm{P\Gamma}L({}_RR^n) = 1.$$

- G is a matched product over all isotypical subgroups!

Open questions

Two of my favorite questions

- Question 1: Can every lattice $L(K, n)$ appear as the strong order interval of a modular quasi-Garside group?
- Question 2: What about the subspace lattice of a non-desarguesian plane?

- Only known answers to Question 1: $L(K, n)$ is realizable if there is an anisotropic hermitian form on K^n or if there is a cyclic field extension L/K of degree n.
  In particular, all finite desarguesian geometries $L(\mathbb{F}_q, n)$ are strong order intervals!
- No answers to Question 2 are known: even for the Hughes plane (91 points and lines), a computational approach fails (even with an insane amount of RAM!).

☺ Thanks for your attention! ☺