Stability and ω-Categoricity of Skew Braces

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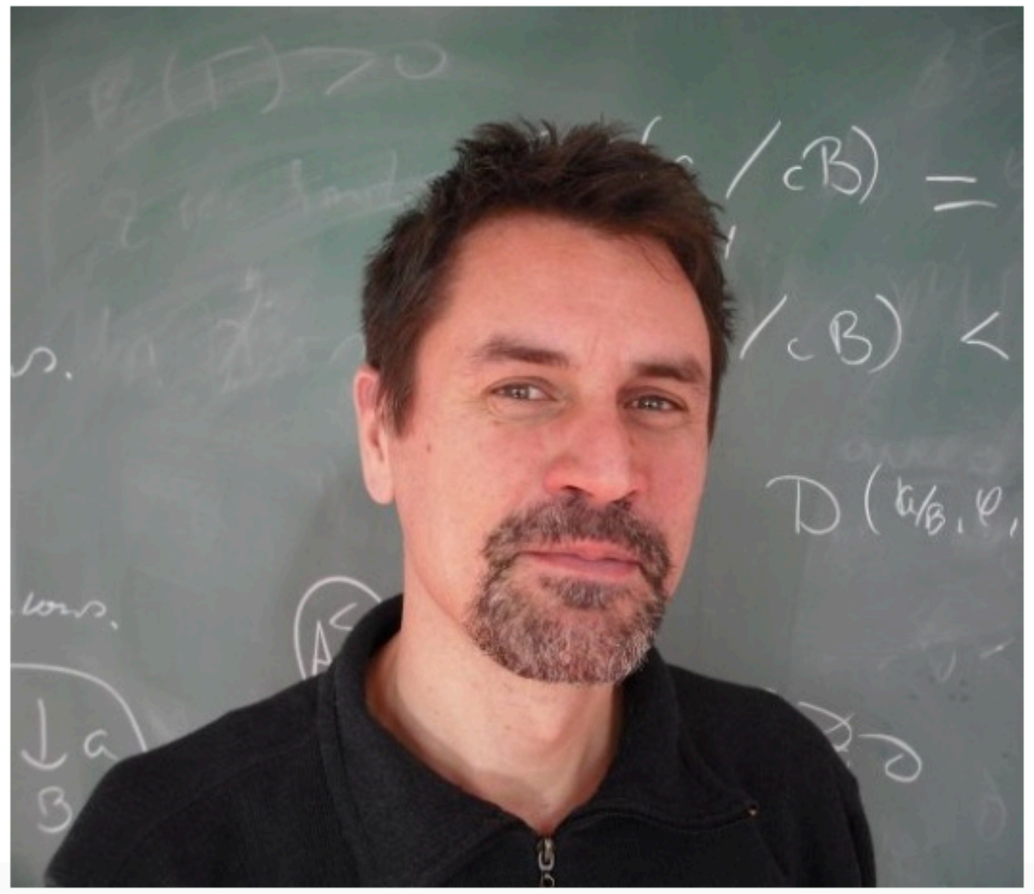

Frank O. Wagner, Université de Lyon, France

Timeline

September 2022

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On derived-indecomposable solutions of the Yang--Baxter equation (I. Colazzo, M.F., M. Trombetti)

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...Now

First-order logic

A vocabulary (or alphabet) τ is a set consisting of relation symbols, function symbols and constant symbols.

First-order logic

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constant

For example, in a group we use the vocabulary $\tau = \{\ \cdot\ , e, \ ^{-1}\}$.

functions

First-order logic

Now, fix a vocabulary τ .

A structure A for τ (a τ -structure) is a non-empty set A together with

- relations $R^A \subseteq A^n$ for every n-ary relation symbol R $\in$ τ ,
- functions $f^A : A^m \to A$ for every m-ary function symbol f $\in$ τ ,
- constants $c^A \in A$ for every constant symbol c $\in$ τ .

Example

If we use the vocabulary $\tau = \{\,\cdot\,, e,\ ^{-1}\}$, we can say that a group is a τ-structure G satisfying the following sentences:

- **(G1)** $\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$
- **(G2)** $\forall x(e \cdot x = x \wedge x \cdot e = x)$
- **(G3)** $\forall x(x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e)$

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Example

An ordered ring R is a structure on the vocabulary $\tau = \{0, 1, +\,, \cdot\,, <\,\}$,

where

- $0{,}1$ are constants,
- $+, \cdot$ are functions,
- $<$ is a relation,

and they are interpreted in accordance with the axioms of the ordered rings.

First-order logic

Before defining formulas, let us define terms.

A term over τ is a finite sequence of characters that can be obtained by finitely many applications of the following rules:

First-order logic

(T1) All constant symbols in τ and all variables are terms.

(T2) If $t_1, \ldots, t_n$ are terms and f $\in \tau$ is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term.

First-order logic

A first-order formula over τ is a finite sequence of characters that can be obtained by finitely many applications of the following rules:

First-order logic

(F1) If t_1 and t_2 are terms over τ, then $(t_1 = t_2)$ is a formula.

(F2) If R $\in \tau$ is an n-ary relation symbol and if $t_1, \ldots, t_n$ are terms over τ,

then $R(t_1, \ldots, t_n)$ is a formula.

(F3) If φ is a formula, then so is $\neg\varphi$.

(F4) If φ and ψ are formulas, then $(\varphi \vee \psi)$ is a formula.

(F5) If φ is a formula and x is a variable, then $\exists x\varphi$ is a formula.

First-order logic

If φ and ψ are formulas, we use

$(\varphi \wedge \psi)$ as abreviations for $\neg(\neg\varphi \vee \neg\psi)$,

$(\varphi \rightarrow \psi)$ as abreviations for $(\neg\varphi \vee \psi)$,

$(\varphi \leftrightarrow \psi)$ as abreviations for $\neg(\neg(\neg(\varphi \vee \psi) \vee \neg(\varphi \vee \neg\psi))$,

$\forall x\varphi$ as abreviations for $\neg\exists x\neg\varphi$.

First-order logic

A variable x occurs **freely** in φ if x occurs outside the scope of a quantifier $\exists x$ or $\forall x$.

Example : $\forall y\,(y=0) \rightarrow (x=0)$

First-order logic

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A formula without free variables is a sentence.

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A variable x occurs freely in φ if x occurs outside the scope of a quantifier $\exists x$ or $\forall x$.

A formula without free variables is a sentence.

A formula is atomic if it contains no quantifiers or logical connectives $\neg, \vee$.

First-order logic

Note: The terms and the formulas haven't meaning if we don't specify a structure in which interpret them.

$v: \{x_1, \ldots, x_n\} \mapsto v(x_i) = a_i \in A$

Now, for every formula $\varphi(x_1, \ldots, x_n)$ and all $a_1, \ldots, a_n \in A$ we define the validity of $\varphi(a_1, \ldots, a_n)$ in A:

First-order logic

Now, for every formula $\varphi(x_1, \ldots, x_n)$ and all $a_1, \ldots, a_n \in A$ we define the validity of $\varphi(a_1, \ldots, a_n)$ in A:

- If $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are formulas, then $(\varphi \vee \psi)(a_1, \ldots, a_n)$ holds in A if and only if at least one of $\varphi(a_1, \ldots, a_n)$ and $\psi(a_1, \ldots, a_n)$ holds in A.

First-order logic

- If $\varphi(x_1, \ldots, x_n)$ is a formula, then $\neg\varphi(a_1, \ldots, a_n)$ holds in A if and only if $\varphi(a_1, \ldots, a_n)$ does not hold in A.

First-order logic

- If $\varphi(x_1, \ldots, x_n)$ is a formula, then $\neg\varphi(a_1, \ldots, a_n)$ holds in A if and only if $\varphi(a_1, \ldots, a_n)$ does not hold in A.
- If $\varphi(x, x_1, \ldots, x_n)$ is a formula, then $\exists x\varphi(a_1, \ldots, a_n)$ holds in A if and only if there is $a \in A$ such that $\varphi(a, a_1, \ldots, a_n)$ holds in A.

First-order logic

If $\varphi(a_1, \ldots, a_n)$ holds in A, we write $A \models \varphi(a_1, \ldots, a_n)$.

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If $\varphi(a_1, \ldots, a_n)$ holds in A, we write $A \models \varphi(a_1, \ldots, a_n)$.

Let Φ be a set of formulas over τ.

If Φ holds in A with respect to every assignment, then we write $A \models \Phi$ and say that A is a model of Φ.

First-order logic

A theory over a vocabulary τ is a set of sentences over τ.

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A theory over a vocabulary τ is a set of sentences over τ.

Given a structure A, the theory of A is the set $Th(A)$ of all sentences φ over τ such that $A \models \varphi$.

Group theory is the theory of the class of all groups.

Skew braces

Let B be a set.

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If $(B, +)$ and $(B, \circ)$ are groups

Skew braces

Let B be a set.

If $(B,+)$ and $(B,\circ)$ are groups, then the triple $(B,+,\circ)$ is a skew (left) brace if the skew (left) distributive property

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds for all $a, b, c \in B$.

Skew braces

Let $(B, +, \circ)$ be a skew (left) brace.

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Recall that in a group $(G, \cdot)$, $[x, y] = xyx^{-1}y^{-1}$

$[a, b]_+$ and $[a, b]_\circ$ will denote respectively the commutator in $(B, +)$ and $(B, \circ)$ of a and b.

Skew braces

The map $\lambda : a \in (B, \circ\,) \mapsto \lambda_a \in Aut(B, +\,)$

where $\lambda_a(b) = -a + a \circ b$

λ is a group homomorphism.

Skew braces

It is possible (and is actually very useful!) to take into account the natural semidirect product

$$G = (B, +) \rtimes (B, \circ)$$

where

$$(a, b)(c, d) = (a + \lambda_b(c), b \circ d)$$

for all $a, b, c, d \in B$.

Skew braces

In analogy with ring theory, a third relevant (non-necessarily associative) operation in skew braces is defined as follows

$$a \star b = \lambda_a(b) - b = -a + a \circ b - b$$

for all $a, b \in B$.

Skew braces

Taking into account $G = (B, +) \rtimes (B, \circ)$, an easy computation shows that the $\star$-operation corresponds to a commutator of type

$$[(0, a), (b, 0)] = (a \star b, 0)$$

for all $a, b \in B$.

Skew brace

A left ideal of a skew brace B is a subgroup I of $(B, +)$ such that $\lambda_a(I) \subseteq I$ for all $a \in B$.

An ideal of a skew brace B is a left ideal that is normal in $(B, +)$ and $(B, \circ)$.

Skew brace

The socle of B is defined as

$$Soc(B) = Ker(\lambda) \cap Z(B, +)$$

Skew brace

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where $Z(B,+)$ is the center of $(B,+)$

Skew brace

The socle of B is defined as

$$Soc(B) = Ker(\lambda) \cap Z(B, +)$$

The annihilator of B is defined as

is the center of $(B, \circ)$

$$Ann(B) = Soc(B) \cap Z(B, \circ)$$

First-order language of skew braces

Let $\tau = \{\, + \,, \circ \,, ^{-1}, - \,, 0\}$ be the first-order language of skew braces.

In what follows, a formula is just an τ-formula, that is, a formula in the language τ.

First-order language of skew braces

$0 = 1$

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Let B be a skew brace.

Then $Th(B) = \{\varphi \,:\, B \vDash \varphi\}$ denotes the first-order theory of B.

First-order language of skew braces

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Let B be a skew brace.

Then $Th(B) = \{\varphi \ : \ B \vDash \varphi\}$ denotes the first-order theory of B.

Hence, Th(B) is the set of all sentences φ over τ such that $B \vDash \varphi$.

First-order language of skew braces

A subset X of B is definable if $X = \{b \in B \ : \ B \vDash \varphi(b)\}$ for some formula $\varphi(x)$.

First-order language of skew braces

A subset X of B is definable if $X = \{b \in B \,:\, B \vDash \varphi(b)\}$ for some formula $\varphi(x)$.

X is parametrically definable if $X = \{b \in B \,:\, B \vDash \varphi(b, a_1, \ldots, a_n)\}$

for some formula $\varphi(x, y_1, \ldots, y_n)$ and some $a_1, \ldots, a_n \in B$,

(in this case, X is also called n-definable).

ω-categorical

A skew brace B is ω-categorical if every countable skew brace which has the same first-order theory as B is isomorphic to B.

ω-categorical

A well-known theorem of Engeler, Ryll–Nardzewski and Svenonius states that B is ω-categorical if and only if, for every $n \in \omega$, $Th(B)$ has only finitely many n-types.

Types of elements

Fix a vocabulary τ.

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Let $n \in \mathbb{N}$ and let $\overline{a} = (a_1, \ldots, a_n) \in M^n$.

Types of elements

Fix a vocabulary τ.

Let M be τ-structure.

Let $n \in \mathbb{N}$ and let $\overline{a} = (a_1, \ldots, a_n) \in M^n$.

The **types** of $\overline{a}$ in M is

$$tp_M(\overline{a}) = \{\varphi(\overline{x}) \ : \ M \vDash \varphi(\overline{a})\}$$

So, this is the set of all formulas $\varphi(\overline{x})$ such that $M \vDash \varphi(\overline{a})$

Types

Let M be τ-structure and let $\bar{x} = (x_1, \ldots, x_n)$ be distinct variables.

A n-types $p(\bar{x})$ of M is a set of formulas over τ, $p(\bar{x}) = \{\varphi_i(\bar{x}) \ : \ i \in I\}$ that is

finitely realized in M.

Types

Let M be τ-structure and let $\overline{x} = (x_1, \ldots, x_n)$ be distinct variables.

A n-types $p(\overline{x})$ of M is a set of formulas over τ, $p(\overline{x}) = \{\varphi_i(\overline{x}) \ : \ i \in I\}$ that is

finitely realized in M.

This means that, for every finite subset $\varphi_1(\overline{x}), \ldots, \varphi_k(\overline{x})$ of $p(\overline{x})$, there exists an element $\overline{a} = (a_1, \ldots, a_n) \in M$ such that $M \vDash \bigwedge_{i \leq k} \varphi_i(\overline{a})$.

Types

If there is $\overline{a}$ in M such that $M \vDash \varphi(\overline{a})$ for all formulas in $\varphi(\overline{x}) \in p(\overline{x})$, we will say that the types $p(\overline{a})$ is realized in M by $\overline{a}$ and we will write $M \vDash p(\overline{a})$.

Types

A type $p(\overline{x})$ is completed, if given any formula $\varphi(\overline{x})$, among the logic implications of $p(\overline{x})$, there is $\varphi(\overline{x})$ or $\neg\varphi(\overline{x})$.

ω-categorical

A well-known theorem of Engeler, Ryll–Nardzewski and Svenonius states that B is ω-categorical if and only if, for every $n \in \omega$, $Th(B)$ has only finitely many n-types.

ω-categorical

THEOREM 2.3.13 (Characterization of ω-Categorical Theories). *Let T be a complete theory. Then the following are equivalent:*

(a). *T is ω-categorical.*

(d). *For each $n < \omega$, T has only finitely many types in $x_1, \ldots, x_n$.*

PROOF. The reader is advised to sit down before beginning this proof. We shall prove the equivalence of (a) and (d) by proving a chain of implications

$$(a) \to (b) \to (c) \to (d) \to (e) \to (f) \to (a).$$

Each of the six equivalent conditions is interesting in its own right.

Model Theory
C.C. Chang
H.J. Keisler

ω-categorical

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$$(a) \to (b) \to (c) \to (d) \to (e) \to (f) \to (a).$$

Each of the six equivalent conditions is interesting in its own right.

m-stable, stable, unstable

Let m be a an infinite cardinal.

m-stable, stable, unstable

Let m be a an infinite cardinal.

The skew brace B is m-stable if and only if, for every subset A of B of cardinality m, the set of complete types over A has cardinality m.

m-stable, stable, unstable

Let m be a an infinite cardinal.

The skew brace B is stable if it is m'-stable for some infinite cardinal m'.

m-stable, stable, unstable

A theory that is not stable is unstable.

Some results

It turns out that an ω-stable skew brace is m-stable for every infinite cardinal m.

Some results

If B is an ω-categorical skew brace, then $(B, +)$ and $(B, \circ)$ have finite exponent.

The exponent of a group is the least natural number n such that

$g^n = 1$, for all $g \in G$.

Some results

If B is an ω-categorical skew brace, then $(B, +)$ and $(B, \circ)$ have finite exponent.

In particular $(B, +) \rtimes_\lambda (B, \circ)$ has finite exponent.

Some results

Let B be a countably infinite ω-categorical skew brace.

Then a subset of B is definable if and only if it is invariant under all automorphisms of B.

Some results

Let B be a skew brace, $N = (B, +)$, $X = (B, \circ)$ and $G = N \rtimes_\lambda X$.

If B is ω-categorical (resp. stable), we easily see that G is ω-categorical (resp. stable) because the function λ is defined in terms of $+$ and $\circ$.

Moreover, it is also clear that both N and X are ω-categorical (resp. stable).

Some results

If B is ω-categorical (resp. stable), then also B/I is ω-categorical (resp. stable) for any definable ideal I of B.

If B is ω-categorical (resp. stable), then also every definable sub-skew brace of B is ω-categorical (resp. stable).

Structural results

The aim of this section is to describe the abstract structure of an arbitrary ω-categorical stable skew brace.

Structural results

Theorem (M.F., M. Trombetti, F. Wagner)

Let B be a ω-categorical skew brace and let $\phi(x_0, x_1, \ldots, x_n)$ be a formula.

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Theorem (M.F., M. Trombetti, F. Wagner)

Let B be a ω-categorical skew brace and let $\phi(x_0, x_1, \ldots, x_n)$ be a formula.

Then there are formulas $\phi^*(x_0, x_1, \ldots, x_n)$ and $\phi^{**}(x_0, x_1, \ldots, x_n)$ such that the following properties hold.

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Let B be a ω-categorical skew brace and let $\phi(x_0, x_1, \ldots, x_n)$ be a formula.

Then there are formulas $\phi^*(x_0, x_1, \ldots, x_n)$ and $\phi^{**}(x_0, x_1, \ldots, x_n)$ such that the following properties hold.

Let $b_1, \ldots, b_n$ and put $T = \{b \in B \; : \; B \models \phi(b, b_1, \ldots, b_n)\}$.

Structural results

Theorem (M.F., M. Trombetti, F. Wagner)

(1) If C is the sub-skew brace generated by T, then

$C = \{b \in B \,:\, B \vDash \phi^*(b, b_1, \ldots, b_n)\}$.

Moreover, if T has finite order n, then C is finite of order $f(n)$ depending only on n.

Structural results

Theorem (M.F., M. Trombetti, F. Wagner)

(2) If I is the ideal generated by T, then $C = \{b \in B \ : \ B \vDash \phi^{**}(b, b_1, \ldots, b_n)\}$.

Structural results

Corollary (M.F., M. Trombetti, F. Wagner)

Let B be an ω-categorical skew brace. Then $B \star B$ is definable.

Some results

A skew brace B is locally-finite if every finitely generated sub-skew brace is finite.

Some results

A skew brace B is locally-finite if every finitely generated sub-skew brace is finite.

Moreover, B is uniformly-locally-finite if there is a function $f : \omega \to \omega$ such that the sub-skew brace generated by n elements has order at most $f(n)$.

Structural results

Corollary (M.F., M. Trombetti, F. Wagner)

Let B be an ω-categorical skew brace. Then B is uniformly-locally-finite.

Applications

A first observation comes from the skew theoretic analog of a group with finitely many conjugates (FC-groups): these skew braces have been introduced and studied in

I. Colazzo – M. F. – M. Trombetti:

On derived-indecomposable solutions of the Yang-Baxter equation

FC-groups

An FC-group is a group in which every element has only finitely many conjugates.

$(G, \cdot)$ is a group. Let $x, y \in G$.

x and y are conjugate if there is $g \in G$ such that $gag^{-1} = b$

Applications

A skew brace B is said to have the property (S) if, for each $b \in B$, there are finitely many elements of the form $b \star c, c \star b, [b,c]_\circ, [b,c]_+$ with $c \in B$.

A skew brace B is said to have the property (BS) if there is $n \in \omega$, such that, for every $b \in B$, there are at most n elements of the form $b \star c, c \star b, [b,c]_\circ, [b,c]_+$ with $c \in B$.

Applications

Theorem (I.Colazzo, M.F., M. Trombetti)

A skew brace has the property (BS) if and only if $B \star B$ and $[B, B]_+$ are finite.

Applications

Theorem (I.Colazzo, M.F., M. Trombetti)

A skew brace has the property (BS) if and only if $B \star B$ and $[B, B]_+$ are finite.

The following is an immediate consequence of the Engeler-Ryll-Nardzewski-Svenonius theorem.

Applications

Theorem (M.F., M. Trombetti, F. Wagner)

Let B be an ω-categorical skew brace. The following are equivalent:

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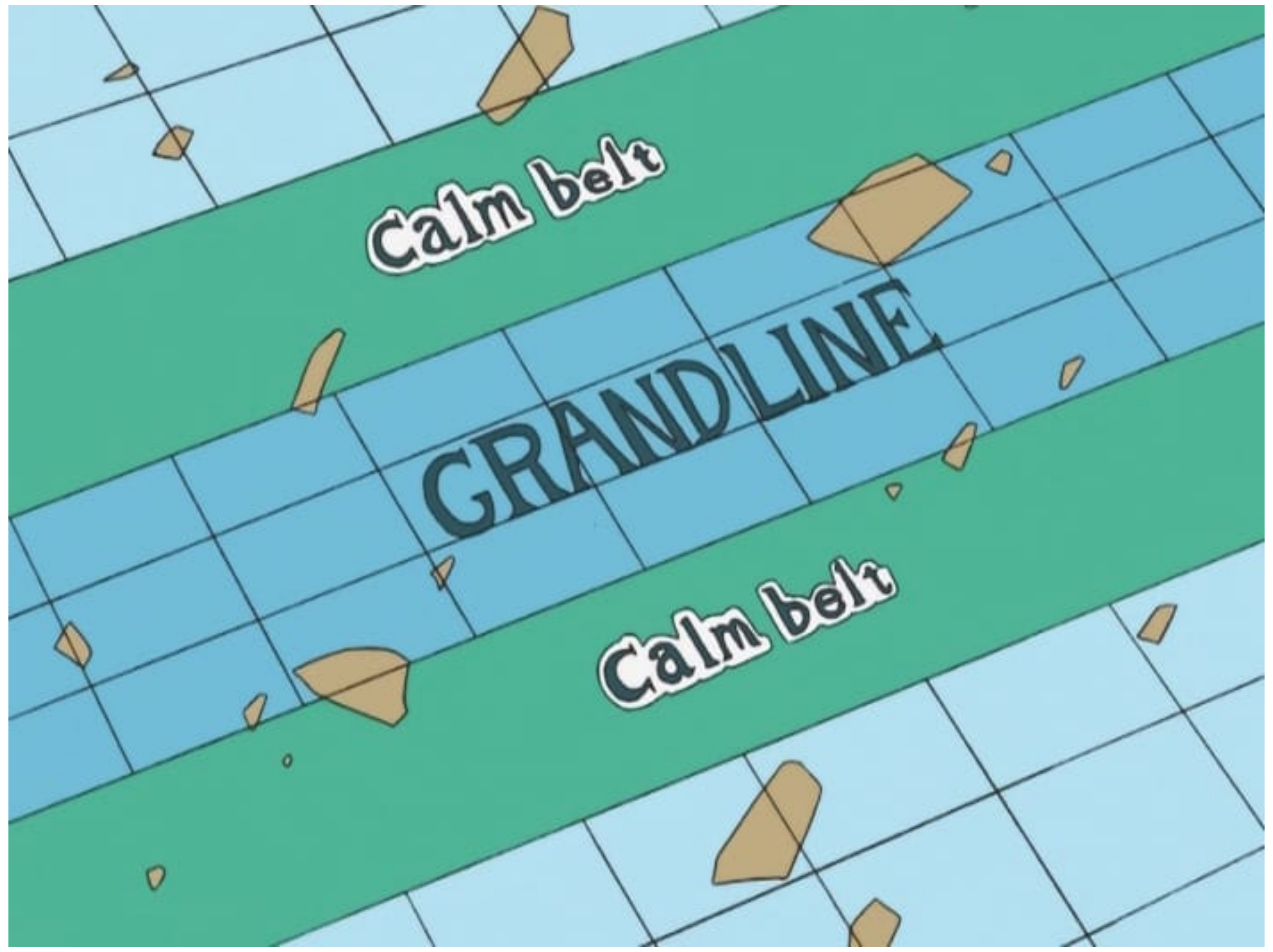

Annihilator-nilpotency

The first nilpotency concept we deal with is annihilator-nilpotency.

Let B be a skew brace.

We define the upper annihilator series of B as follows

Put $Ann_0(B) = \{0\}$; for any ordinal α, let $Ann_{\alpha+1}(B)/Ann_\alpha(B) = Ann(B/Ann_\alpha(B))$.

Annihilator-nilpotency

If ν is a limit ordinal, let $Ann_\nu(B) = \bigcup_{\alpha<\nu} Ann_\alpha(B)$.

The smallest ordinal number $a(B)$ such that $Ann_{a(B)}(B) = Ann_{a(B)+1}(B)$ is the annihilator-length of B.

The last term of the upper annihilator series is the hyper-annihilator of B and is denoted by $\overline{Ann}(B)$.

Annihilator-nilpotency

If $B = Ann_n(B)$ for some $n \in \omega$, we say that B is annihilator-nilpotent.

If $B = \overline{Ann}(B)$ we say that B is annihilator-hypercentral.

Annihilator-nilpotency

If $B = Ann_n(B)$ for some $n \in \omega$, we say that B is annihilator-nilpotent.

If $B = \overline{Ann(B)}$ we say that B is annihilator-hypercentral.

Moreover, B is locally-annihilator-nilpotent if every finitely generated sub-skew brace of B is annihilator-nilpotent.

Nilpotency

The aim of this section is to show that most of time these nilpotency concepts coincide for categorical/stable skew braces.

Applications

Theorem (M.F., M. Trombetti, F. Wagner)

Let B be an ω-categorical, stable skew brace. The following statements are equivalent:

(1) B is locally-annihilator-nilpotent

(2) B is annihilator-nilpotent

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Theorem (M.F., M. Trombetti, F. Wagner)

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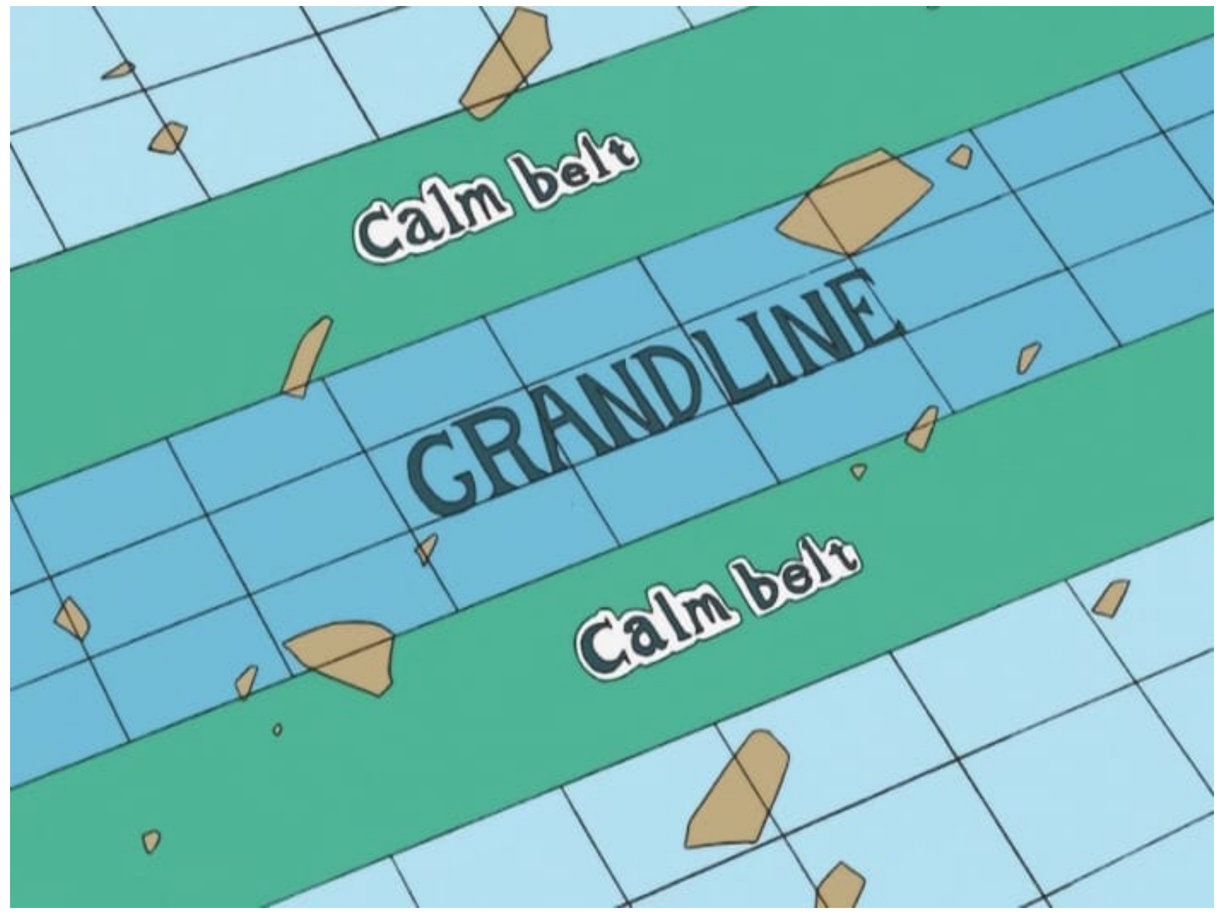

Right nilpotency

Let B be a skew brace and let S, T subsets of B.

Put $R_0(S, T) = S$ and

$$R_n(S, T) = R_{n-1}(S, T) \star T$$

for $n > 0$.

Thus $R_m(S, T)$ is recursively defined for every non-negative integer m.

Right nilpotency

B is right nilpotent if and only if there is some integer c such that $R_c(B, B) = \{0\}$.

Right nilpotency was introduced by Rump for braces.

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B is right nilpotent if and only if there is some integer c such that $R_c(B, B) = \{0\}$.

Right nilpotency was introduced by Rump for braces.

W. Rump

"Braces, radical rings, and the quantum Yang-Baxter equation"

J. Algebra 307 (2007), 153-170.

Right nilpotency

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"Skew left braces of nilpotent type"

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"On skew braces (with an appendix by N. Byott and L. Vendramin)"

J. Comb. Algebra 2 (2018), no. 1, 47-86.

Right nilpotency

Theorem (M.F., M. Trombetti, F. Wagner)

(B,+) is nilpotent

Let B be a ω-categorical, stable skew brace of nilpotent type. Then the following statements are equivalent:

(1) B is right nilpotent

(2) B is locally-right nilpotent

(3) B has a finite s-series

(4) $B = Soc_n(B)$ for some $n \in \omega$

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A skew brace B is said to be locally-right-nilpotent if every finitely generated sub-skew brace of B is right nilpotent.

Left nilpotency

Let B be a skew brace and let X, Y be subsets of B.

Put $L_0(X, Y) = Y$ and

$$L_n(X, Y) = X \star L_{n-1}(X, Y)$$

for $n > 0$.

Thus $L_m(X, Y)$ is recursively defined for every non-negative integer m.

Left nilpotency

B is left nilpotent if and only if there is some integer c such that $L_c(B,B) = \{0\}$.

Left nilpotency

F. Cedò, A. Smoktunowicz, L. Vendramin

"Skew left braces of nilpotent type"

Proc. London Math. Soc. (6) 118 (2019), 1367-1392.

For a finite skew brace B of nilpotent type, being left nilpotent is equivalent to $(B, \circ)$ being nilpotent.

Here, we extend this result to ω-categorical, stable skew brace.

Left nilpotency

Theorem (M.F., M. Trombetti, F. Wagner)

Let B be a ω-categorical, stable skew brace of nilpotent type.

If C is any sub-skew brace of B, the following statements are equivalent:

(1) C is left nilpotent

(2) $(C, \circ\,)$ is nilpotent

Left nil

A skew brace B is left nil if for every $b \in B$ there is $n \in \omega$ such that

$$\underbrace{b \star (b \star (\ldots \star b)\ldots))}_{n \quad times} = 0$$

Left nil

A. Smoktunowicz

"A note on set-theoretic solutions of the Yang-Baxter equation"

J. Algebra 500 (2018), 3-18.

Smoktunowicz proved that if B is finite and $(B, +)$ is abelian, then B is left nil if and only if it is left nilpotent.

Left nilpotency

Theorem (M.F., M. Trombetti, F. Wagner)

Let B be a ω-categorical, stable brace. If C is any sub-skew brace of B, the following statements are equivalent:

(1) C is left nil

(2) C is left nilpotent

Thank you for listening!