The Quiver-Theoretic Dynamical Yang–Baxter Equation

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Groups, Rings, and the Yang–Baxter Equation
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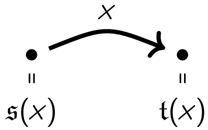
Let Λ be a nonempty set.

A **quiver** Q **over** Λ is a directed graph, with set of vertices Λ.

We denote by Q both the quiver and the set of *edges* (or "*arrows*").

We denote by $\mathfrak{s}, \mathfrak{t} \colon Q \to \Lambda$ the ***source*** and ***target*** maps.

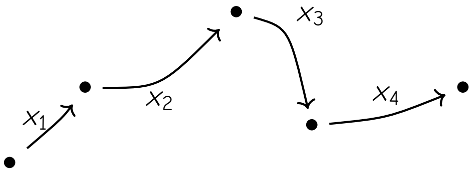


A **map of quivers over** Λ is a map between the sets of edges, which preserves the sources and the targets.

A path $x_1|\dots|x_s$ in Q is a sequence of *composable edges*, i.e.,

$$\mathfrak{s}(x_{i+1}) = \mathfrak{t}(x_i).$$

E.g., a path of length 4:



$\mathrm{Path}(Q)$ has a natural structure of a **category**, with the **junction of paths**.

Identities: id_λ is given by ε_λ, the *empty path* on λ.

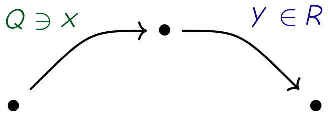
Let Q, R be quivers over **the same** Λ.

The product $Q \otimes R$ is defined as follows:

$$Q \otimes R = \{(x, y) \mid x \in Q, y \in R, \mathfrak{s}_R(y) = \mathfrak{t}_Q(x)\}.$$

$$\mathfrak{s}_{Q \otimes R}(x, y) = \mathfrak{s}_Q(x), \quad \mathfrak{t}_{Q \otimes R}(x, y) = \mathfrak{t}_R(y).$$

In other words, $Q \otimes R$ is the set of all paths



$Q \otimes Q$ is naturally identified with $\mathrm{Path}_2(Q)$.
In the same way, $Q^{\otimes n}$ is naturally identified with $\mathrm{Path}_n(Q)$.

Let $\mathscr{A}$ be a quiver, $\sigma\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ a map of quivers over Λ.

Definition

We say that σ is a solution to the **dynamical Yang-Baxter Equation**, or that σ is a **dynamical Yang-Baxter map**, if it satisfies the following **braid relation**:

$$(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma).$$

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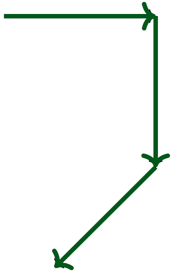

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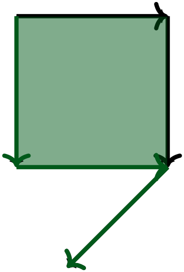

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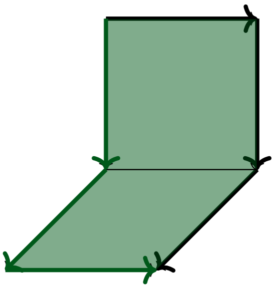

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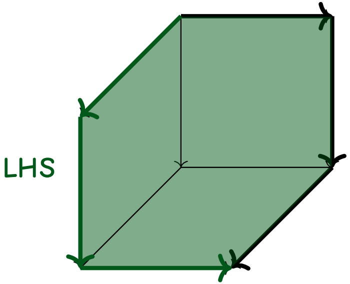

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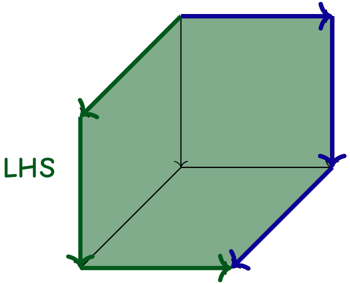

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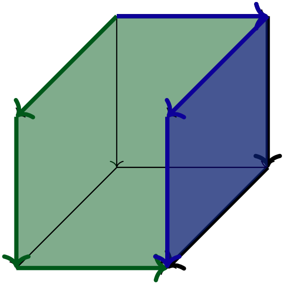
LHS



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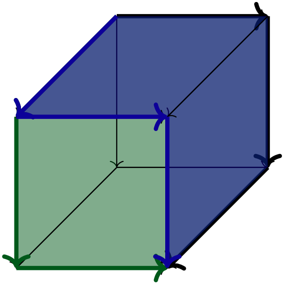

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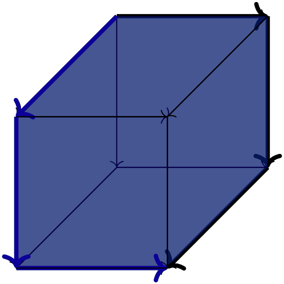
LHS

=

RHS



Definition (Shibukawa, 2005)

A **dynamical set** (or *d-set*) over a nonempty Λ is the datum of a set X and a ***transition map*** $\phi\colon \Lambda \times X \to \Lambda$.

There exists a notion of **maps of d-sets over** Λ. The d-sets over Λ form a category DSet_Λ.

There exists a notion of **tensor product** $\otimes$ for d-sets over Λ.

$(\mathrm{DSet}_\Lambda, \otimes)$ is **monoidal**.

Definition (Shibukawa, 2005)

Let X be a d-set over Λ. A map of d-sets $\sigma\colon X \otimes X \to X \otimes X$ is a dynamical YB map if and only if it satisfies

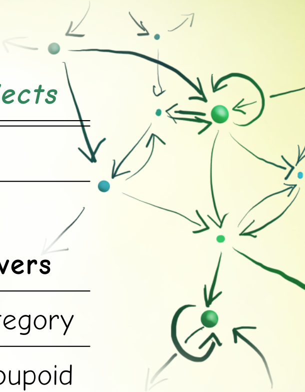
$$\sigma_{12}(\lambda)\sigma_{23}(\phi(\lambda, X^{(1)}))\sigma_{12}(\lambda) = \sigma_{23}(\phi(\lambda, X^{(1)}))\sigma_{12}(\lambda)\sigma_{23}(\phi(\lambda, X^{(1)})).$$

There is a fully faithful strong monoidal functor $\mathcal{Q}\colon \mathsf{DSet}_\Lambda \to \mathsf{Quiv}_\Lambda$.

Via this functor,

dynamical YB map σ on a d-set $X \longmapsto$ braiding $\mathcal{Q}(\sigma)$ on the quiver $\mathcal{Q}(X)$.

One object	*Multiple objects*
Sets	Quivers
YBE: **Braided sets**	DYBE: **Braided quivers**
Structure Monoid	Structure Category
Structure Group	Structure Groupoid
⋮	⋮



DYBE is a **very natural generalisation of YBE:** we expect to get a **more general theory**, just **as strong as the YBE theory**, with very **little effort**.

Structure category and structure groupoid

Let σ be a DYBM on $\mathscr{A}$, and denote $\sigma(x, y) = (x \rightharpoonup y, x \leftharpoonup y)$.

Definition

The **structure category** $\mathscr{C}(\sigma)$ of σ is

$$\mathscr{C}(\sigma) = \langle \mathscr{A} \mid x|y \sim (x \rightharpoonup y)|(x \leftharpoonup y) \rangle^{+},$$

the category $\mathrm{Path}(\mathscr{A})$ modulo the relations $x|y \sim \sigma(x|y)$.

The **structure groupoid** $\mathscr{G}(\sigma)$ is defined as

$$\mathscr{G}(\sigma) = \langle \mathscr{A} \mid x|y \sim (x \rightharpoonup y)|(x \leftharpoonup y) \rangle,$$

the category $\mathrm{Path}(\mathscr{A} \cup \mathscr{A}^{\mathrm{op}})$ modulo the relations $x|y \sim \sigma(x|y)$, $x x^{\mathrm{op}} \sim \mathrm{id}$, $x^{\mathrm{op}} x \sim \mathrm{id}$.

Garside structure

Theorem (Chouraqui, 2010)

Suppose σ is an ***involutive non-degenerate*** *YB map.*

Then, the ***structure monoid*** *$M(\sigma)$ is Garside.*

Moreover, $M(\sigma)$ embeds in the ***structure group*** *$G(\sigma)$, which is the same as the* ***fraction group*** *of $M(\sigma)$, and hence inherits the Garside structure.*

A **Garside category** is a category equipped with a special **normal form**.

This normal form allows us to solve the **word problem**, in a way which can be **computationally efficient**.

Garside structure (dynamical case)

We may expect this to generalise to the dynamical case. This is indeed what happens:

Theorem (F.–Shibukawa, 2023)

Suppose σ is an ***involutive non-degenerate*** *dynamical YB map.*

Then, the structure category $\mathscr{C}(\sigma)$ is Garside.

The category $\mathscr{C}(\sigma)$ embeds in its enveloping groupoid $\mathrm{Env}(\mathscr{C}(\sigma))$, which inherits the Garside structure. Moreover, $\mathscr{G}(\sigma)$ and $\mathrm{Env}(\mathscr{C}(\sigma))$ are the same groupoid.

Preliminaries to dynamical braces

Definition

A **right quasigroup** is a set A with a binary operation $\circ$ such that all the right-multiplication maps

$$\_ \circ a \colon A \to A$$

$$b \mapsto b \circ a,$$

for all $a \in A$, are bijections.

Notice that there is no request for associativity.

A **brace** is the datum of a set A, and of **two group structures** $(A,+)$, $(A,\circ)$ on A, with $(A,+)$ abelian, such that the following **compatibility relation** is satisfied:

$$a \circ (b+c) = a \circ b - a + a \circ c$$

for all $a, b, c \in A$.

Definition (D. K. Matsumoto, 2013)

A ***dynamical brace*** (or ***d-brace***) over Λ is the datum of a **dynamical set** (A,ϕ), an **abelian group** structure $(A,+)$ on A, and a family of **right-quasigroup structures** $\{(A,\circ_\lambda)\}_{\lambda\in\Lambda}$ on A, such that the following **dynamical associativity** holds:

$$a \circ_\lambda (b \circ_\lambda c) = (a \circ_{\phi(\lambda,c)} b) \circ_\lambda c,$$

and such that the following **compatibility relation** holds:

$$a \circ_\lambda (b+c) = a \circ_\lambda b - a + a \circ_\lambda c$$

for all $\lambda \in \Lambda$, $a, b, c \in A$.

Theorem (Matsumoto, 2013)

From each ***dynamical brace*** *of abelian type, one can construct an involutive non-degenerate* ***dynamical YB map****.*

These are originally constructed as dynamical YB maps on the **d-set** (A, ϕ).

$$\xrightarrow{\quad \mathcal{Q} \quad} \textbf{braided quiver}.$$

However, it is not immediately clear how the corresponding quivers look like.

How do we construct the quiver of a dynamical brace?

Let A be a d-brace. We can associate with it a **family** $\mathscr{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ of **subsets of the holomorph** $S_\lambda \subset A \rtimes \mathrm{Aut}(A)$, as follows:

$$S_\lambda = \{(a, \gamma_a^\lambda) \mid a \in A\}$$

with $\gamma_a^\lambda \colon b \mapsto b \circ_\lambda a - a$. Notice that γ_a^λ lies in $\mathrm{Aut}(A, +)$ for all λ, a.

The S_λ's are the vertices of the quiver.

The **edges** are obtained as follows: for each $(a, f) \in S_\lambda$ such that $(a, f)^{-1} S_\lambda = S_\mu$, we draw an edge

$$S_\lambda \longrightarrow S_\mu.$$

Definition

A d-brace A is ***symmetric*** if all the S_λ's contain the pair $(0, \mathrm{id}_A)$. Equivalently, if all the vertices are looped.

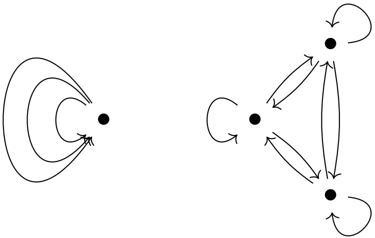
In order to classify the quivers of dynamical braces, it suffices to consider the ***maximal symmetric* dynamical braces**.

Given an abelian group $(A, +)$, the maximal symmetric dynamical brace on A is **uniquely determined**.

Theorem (F.–Shibukawa, 2023)

The quiver of a symmetric dynamical brace is a disjoint union of ***complete quivers****.*

The cardinality of each connected component divides the order of A.



The quiver of the maximal symmetric d-brace on $A = \mathbb{Z}/3\mathbb{Z}$.

With each $d | o(A)$, we may associate the **number n_d of connected components of cardinality d.**

Proposition (F.–Shibukawa, 2023)

For $A = \mathbb{Z}/p\mathbb{Z}$, we have

$$n_1 = p, \qquad n_p = (p-1)^{p-1} - 1.$$

Thank you for your attention.