The modular isomorphism problem and abelian direct factors

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Groups, Rings, and the Yang-Baxter equation.
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The modular isomorphism problem

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- G, H finite p-groups.
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In its extreme form,

Question (MIP)

Can the isomorphism type of G be recovered from $\mathbb{F}_p G$ (resp. kG)?

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- $G/\mathcal{M}_4(G)$ if $p > 2$ (Hertweck).

Direct product decompositions

Let

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Question

If $\mathbb{F}_p G \cong \mathbb{F}_p H$, then

$$H = H_1 \times H_2 \times \cdots \times H_n,$$

with each H_i indecomposable, and

$$\mathbb{F}_p G_i \cong \mathbb{F}_p H_i \quad (\textit{for each } i)?$$

Direct product decompositions

Let

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where $\mathrm{El}(G)$ is elementary abelian and $\mathrm{NEl}(G)$ has no elementary abelian direct factors.

Theorem (Margolis, Sakurai, Stanojkovski)

$$kG \cong kH \textit{ if and only if } \begin{cases} \mathrm{El}(G) \cong \mathrm{El}(H), \textit{ and} \\ k\,\mathrm{NEl}(G) \cong k\,\mathrm{NEl}(H). \end{cases}$$

Direct product decompositions

Let

$$G = \mathrm{Ab}(G) \times \mathrm{NAb}(G),$$

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Theorem (GL)

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Related questions

Clearly $k(G \times H) \cong kG \otimes_k kH$.

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If G is an indecomposable finite p-group, then is kG indecomposable as tensor product of k-algebras?

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If G is an indecomposable finite p-group, then is kG indecomposable as tensor product of k-algebras?

In the commutative case, this is true:

Theorem (Carlson-Kovacs)

Let G be a finite abelian p-group. If $kG = A_1 \otimes A_2 \otimes \cdots \otimes A_n$ then

$$G = G_1 \times \cdots \times G_n$$

and

$$kG_i \cong A_i \qquad (\textit{for each } i).$$

Ingredients of the proof of

Theorem (GL)

$$\mathbb{F}_p G \cong \mathbb{F}_p H \textit{ if and only if } \begin{cases} \mathrm{Ab}(G) \cong \mathrm{Ab}(H), \ \textit{and} \\ \mathbb{F}_p \mathrm{NAb}(G) \cong \mathbb{F}_p \mathrm{NAb}(H). \end{cases}$$

Ingredients of the proof: lattices of normal subgroups

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- R ring, G finite group.
- If $N \trianglelefteq G$, then $I(RN)RG$ denotes the ideal of RG generated by $\{n-1 : n \in N\}$.
- Let $\mathcal{L}$ be a "rule" that assigns $G \mapsto \mathcal{L}_G$, a sublattice of the lattice of normal subgroups of G.
- $\mathcal{L}$ is canonical if for each isomorphism $\phi : RG \to RH$ there is an isomorphism of lattices $\bar{\phi} : \mathcal{L}_G \to \mathcal{L}_H$ such that

$$\phi(I(RN)RG) = I(R\bar{\phi}(N))RH \qquad \text{for each } N \in \mathcal{L}_G.$$

Canonical lattices of normal subgroups: examples

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- For $R = k$, and G a finite p-group, and $\mathcal{L}_G$ defined as follows:

$$\mathcal{S}_0 = \{G'\},$$
$$\mathcal{S}_{i+1} = \mathcal{S}_i \cup \left\{ \begin{array}{c} \Omega_t(G:N), \\ \mho_t(L)N, \\ \Omega_t(\mathcal{Z}(G))N \end{array} \middle| N, L \in \mathcal{S}_i, G' \subseteq N \right\},$$
$$\mathcal{L}_G = \bigcup_{i \geq 1} \mathcal{S}_i.$$

Then $\mathcal{L}_G$ is canonical.

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And the other way around?

- Not true in general: if G and H are finite abelian groups,

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Proposition (GL-del Río)

F any field, G and H finite groups. If $FG \cong FH$, then there is a finite extension F_0 of the prime field of F such that $F_0 G \cong F_0 H$.

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Thanks for your attention.