# Segre products and Segre morphisms in a class of Yang–Baxter algebras

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## 1. "A solution of YBE" = "a solution" = "a nondegenerate involutive set-theoretic solution $(X, r)$ of YBE"

- ***The Yang-Baxter algebras*** $\mathcal{A}_X = \mathcal{A}(K, X, r)$ related to solutions $(X, r)$, of finite order $|X| = n$ over a field $K$ will play a central role in the talk.
- It was proven in [GIVB 98] that the quadratic algebra $\mathcal{A}_X$ of every finite solution $(X, r)$ of YBE has remarkable algebraic, homological and combinatorial properties. In general, the algebra $\mathcal{A}_X$ is noncommutative and in most cases it is not even a PBW algebra, but it preserves various good properties of the commutative polynomial ring $K[x_1, \cdots, x_n]$:
- $\mathcal{A}_X$ has finite global dimension and polynomial growth,
- $\mathcal{A}_X$ is Cohen-Macaulay, Koszul, and a Noetherian domain.
- In the special case when $(X, r)$ is a square-free solution $\mathcal{A}_X$ is a PBW Artin-Schelter regular algebra.

- The study of non-commutative algebras defined by quadratic relations as examples of *quantum non-commutative spaces* has received considerable impetus from the seminal work of Faddeev, Reshetikhin and Takhtajan,1989, and from Manin's Programme for non-commutative geometry, 1991.
- Following Manin (Quantum Groups, 1988) we call the quadratic algebras related to set-theoretic solutions of the Yang-Baxter equation ***Yang-Baxter algebras*** (GI, 2004).
- The YB algebras we study are important for both noncommutative algebra and non-commutative algebraic geometry, as they provide a rich source of examples of interesting associative algebras and non-commutative spaces some of which are Artin-Schelter regular algebras.

# Main Problem

Let $(X, r_X)$ and $(Y, r_Y)$ be finite solutions of YBE whose Yang-Baxter algebras are $A = \mathcal{A}(K, X, r_X)$ and $B = \mathcal{A}(K, Y, r_Y)$, respectively.

(1) Find a presentation of the Segre product $A \circ B$ in terms of one-generators and linearly independent quadratic relations.

(2) Introduce analogues of Segre maps for the class of Yang-Baxter algebras of finite solutions of YBE.

(3) Study separately Segre products and Segre maps in the special case when $(X, r_X)$ and $(Y, r_Y)$ are square-free solutions.
Note that only in this case the algebras $A$ and $B$ are PBW (binomial skew polynomial rings).

Our approach is entirely algebraic and combinatorial. The problem is solved completely.

# 4. Segre products of graded algebras

**Definition.** Let

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots \quad \text{and} \quad B = B_0 \oplus B_1 \oplus B_2 \oplus \cdots$$

be $N_0$- graded algebras over a field $K$, where $K = A_0 = B_0$ and $N_0$ is the set of non-negative integers.
The *Segre product* of $A$ and $B$ is the $N_0$-graded algebra

$$A \circ B := \bigoplus_{i \geq 0} (A \circ B)_i \text{ with } (A \circ B)_i = A_i \otimes_K B_i.$$

$A \circ B$ is a subalgebra of $A \otimes B$.
The embedding is not a graded algebra morphism, as it doubles grading.

# Skip*

If $A$ and $B$ are locally finite then the Hilbert function of $A \circ B$ satisfies

$$\begin{aligned} h_{A\circ B}(t) &= \dim(A \circ B)_t = \dim(A_t \otimes B_t) \\ &= \dim(A_t) \cdot \dim(B_t) = h_A(t) \cdot h_B(t). \end{aligned}$$

Moreover, $A \circ B$ inherits various properties from the two algebras $A$ and $B$. In particular, if both algebras are one-generated, quadratic, and Koszul, then the algebra $A \circ B$ is also one-generated, quadratic, and Koszul.

# The quadratic relations of $A \circ B$ (the general case)*

Suppose that $A$ and $B$ are quadratic algebras generated in degree one by $A_1$ and $B_1$, resp., written as:

$$A = T(A_1)/(\Re_A) \qquad \text{with } \Re_A \subset A_1 \otimes A_1,$$
$$B = T(B_1)/(\Re_B) \qquad \text{with } \Re_B \subset B_1 \otimes B_1,$$

where $T(-)$ is the tensor algebra and $(\Re_A), (\Re_B)$ are the ideals of relations of $A$ and $B$.
Then $A \circ B$ is also a quadratic algebra generated in degree one by $A_1 \otimes B_1$ and presented as

$$A \circ B = T(A_1 \otimes B_1)/(s^{23}(\Re_A \otimes B_1 \otimes B_1 + A_1 \otimes A_1 \otimes \Re_B)),$$

where

$$s^{23}(a_1 \otimes a_2 \otimes b_1 \otimes b_2) = a_1 \otimes b_1 \otimes a_2 \otimes b_2.$$

## 5. The Yang-Baxter algebras

Let $(X, r_1)$ be a solutions of YBE, $|X| = m$. Fix an enumeration $X = \{x_1, \cdots, x_m\}$ and extend it to deg-lex order on the free monoid $\langle X \rangle$. The ***Yang-Baxter algebra*** $A = \mathcal{A}(K, X, r_1)$ is defined as

$$A = K\langle X\rangle/(\Re_1), \text{ where } \Re_1 \text{ is a set of } \binom{m}{2} \text{ binomial relations :}$$
$$\Re_1 = \{x_j x_i - x_{i'} x_{j'} \mid r_1(x_j x_i) = x_{i'} x_{j'}, \text{ and } x_j x_i > x_{i'} x_{j'}\}.$$

$A$ is a f.p. quadratic algebra naturally graded by length:
$A = A_0 \oplus A_1 \oplus A_2 \oplus, A_0 = K, A_1 = \text{Span}X, \cdots.$
Let $(Y, r_2)$ be a solution, with $Y = \{y_1, \cdots, y_n\}$. Similarly, we extend the enumeration to deg-lex oreder on $\langle Y \rangle$. The YB-algebra $B = \mathcal{A}(K, Y, r_2)$ is defined as

$$B = K\langle Y\rangle/(\Re_2), \text{ where } \Re_2 \text{ is a set of } \binom{n}{2} \text{ binomial relations :}$$
$$\Re_2 = \{y_b y_a - y_{a'} y_{b'} \mid r_2(y_b y_a) = y_{a'} y_{b'} \text{ and } y_b y_a > y_{a'} y_{b'}\}.$$

Similarly,$B = B_0 \oplus B_1 \oplus A_2 \oplus, B_0 = K, B_1 = \text{Span}Y, \cdots$ is a quadratic graded algebra.

# 6. Problem 1.

Find a finite presentation of the Segre product $A \circ B$ *in terms of one-generators and linearly independent quadratic relations*. Recall that

$$A \circ B := \bigoplus_{i \geq 0} (A \circ B)_i \text{ with } (A \circ B)_i = A_i \otimes_K B_i.$$

# Skip*

## Remark

*In general, $\Re_1$ and $\Re_2$ are not necessarily relations of binomial skew polynomial algebras.*

One has

$$\dim A_2 = \binom{m+1}{2}, \quad \dim B_2 = \binom{n+1}{2},$$

$$\dim (A \circ B)_2 = \binom{m+1}{2}\binom{n+1}{2}.$$

# 7. The Cartesian product of braided sets

**Definition.** Let $(X, r_1)$ and $(Y, r_2)$ be disjoint braided sets (we do not assume involutiveness, nor nondegeneracy). Consider the Cartesian product of sets $X \times Y$ and the bijective map

$$\mathfrak{r} : (X \times Y) \times (X \times Y) \longrightarrow (X \times Y) \times (X \times Y) \text{ defined as}$$
$$\mathfrak{r} := s_{23} \circ (r_1 \times r_2) \circ s_{23},$$

where $s_{23}$ is the flip of the second and the third component. In other words,

$$\mathfrak{r}((x_j, y_b), (x_i, y_a)) := (({}^{x_j}x_i, {}^{y_b}y_a), (x_j^{x_i}, y_b^{y_a})),$$

for all $i, j \in \{1, \cdots, m\}$ and all $a, b \in \{1, \cdots, n\}$. Then the quadratic set $(X \times Y, \mathfrak{r})$ is a braided set of order $mn$, and we shall refer to it as
***the Cartesian product of the braided sets*** $(X, r_1)$ ***and*** $(Y, r_2)$.

## 8. The Cartesian product of braided sets, $(X \times Y, \mathfrak{r})$ satisfies the following conditions.

- $(X \times Y, \mathfrak{r})$ is nondegenerate *iff* $(X, r_1)$ and $(Y, r_2)$ are nondegenerate.
- $(X \times Y, \mathfrak{r})$ is involutive *iff* $(X, r_1)$ and $(Y, r_2)$ are involutive.
- $(X \times Y, \mathfrak{r})$ is a solution of YBE *iff* $(X, r_1)$ and $(Y, r_2)$ are solutions of YBE.
- $(X \times Y, \mathfrak{r})$ is a square-free solution *iff* $(X, r_1)$ and $(Y, r_2)$ are square-free solutions.

9. Let $(X, r_1)$ and $(Y, r_2)$ be solutions on the disjoint sets $X = \{x_1, \cdots, x_m\}$, and $Y = \{y_1, \cdots, y_n\}$.

$A \circ B$ is the Segre product of the YB algebras $A = \mathcal{A}(K, X, r_1)$ and $B = \mathcal{A}(K, Y, r_2)$. To simplify notation we write "$x \circ y$" instead of "$x \otimes y$", $x \in X, y \in Y$, or "$u \circ v$" instead of "$u \otimes v$", for $u \in A_d, v \in B_d, d \geq 2$. Let

$$X \circ Y = \{x_i \circ y_a \mid 1 \leq i \leq m,\ 1 \leq a \leq n\}.$$

**Proposition-Notation.** There is a natural structure of a solution $(X \circ Y, \mathbf{r}_{X \circ Y})$ given by

$$\mathbf{r}_{X \circ Y}((x_j \circ y_b), (x_i \circ y_a)) := ((({}^{x_j}x_i) \circ ({}^{y_b}y_a)), ((x_j^{x_i}) \circ (y_b^{y_a}))),$$

$1 \leq i, j \leq m, 1 \leq a, b \leq n$.
This solution is isomorphic to the Cartesian product of solutions $(X \times Y, \mathfrak{r})$. In particular, $(X \circ Y, r_{X \circ Y})$ has cardinality $mn$ and $\binom{mn}{2}$ nontrivial $r_{X \circ Y}$-orbits.

# Skip*

$(X \circ Y, \mathbf{r})$ has exactly $mn$ fixed points, namely:

$$\mathcal{F} = \{(x_p \circ y_a)(x_q \circ y_b) \mid r_1(x_p x_q) = x_p x_q, \text{ and } r_2(y_a y_b) = y_a y_b, \text{ where } p, q \in \{1, \cdots, m\},\ a, b \in \{1, \cdots, n\}\}.$$

In this case $x_p x_q \in \mathcal{N}(A)_2$ and $y_a y_b \in \mathcal{N}(B)_2$.

# Skip*

**Proposition.** The YB algebra $\mathfrak{A} = \mathfrak{A}_{X \circ Y} = \mathcal{A}(K, X \circ Y, r)$ is generated by the set $X \circ Y$ and has $\binom{mn}{2}$ quadratic defining relations described in the two lists below.

$$(1) \quad f_{ji,ba} = (x_j \circ y_b)(x_i \circ y_a) - ({}^{x_j}x_i \circ {}^{y_b}y_a)(x_j^{x_i} \circ y_b^{y_a}),$$
$$\text{for all } \ 1 \leq i, j \leq m \ \text{ s.t. } \ x_j > {}^{x_j}x_i,$$
$$\text{and all } \ 1 \leq a, b \leq n.$$

The leading monomials are $LM(f_{ji,ba}) = (x_j \circ y_b)(x_i \circ y_a)$.

$$(2) \quad f_{ij,ba} = (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ {}^{y_b}y_a)(x_j \circ y_b^{y_a}),$$
$$\text{for all } \ 1 \leq i, j \leq m \ \text{ with } \ r_1(x_i x_j) = x_i x_j,$$
$$\text{and all } \ 1 \leq a, b \leq n, \ \text{ s. t. } \ y_b > {}^{y_b}y_a.$$

The leading monomials are $LM(f_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a)$.

## 10 Corollary

Let $(X, r_1)$ and $(Y, r_2)$ be finite solutions and let $A = \mathcal{A}(K, X, r_1)$ and $B = \mathcal{A}(K, Y, r_2)$ be their Yang-Baxter algebras. Then the Segre product, $A \circ B$ is a one-generated quadratic and Koszul algebra.
This is a consequence from the results of GIVB (1998) and a proposition in the book "Quadratic Algebras" PoPo

We also prove that $A \circ B$ is *a left and a right Noetherian algebra with polynomial growth*.

## 11. Theorem A

Suppose $(X, r_1)$ and $(Y, r_2)$ are finite solutions,
$X = \{x_1 \cdots, x_m\}$, $Y = \{y_1 \cdots, y_n\}$ are disjoint sets
$A = \mathcal{A}(K, X, r_1)$ and $B = \mathcal{A}(K, Y, r_2)$.
Let $A \circ B$ be the Segre product of $A$ and $B$, and let
$(X \circ Y, r_{X \circ Y})$ be the solution of YBE defined above.

**Theorem A.**
*The algebra $A \circ B$ has a set of $mn$ one-generators $W = X \circ Y$ ordered lexicographically:*

$$\begin{aligned} W = \{w_{11} &= x_1 \circ y_1 < w_{12} = x_1 \circ y_2 < \cdots < w_{1n} = x_1 \circ y_n \\ &< w_{21} = x_2 \circ y_1 < \cdots < w_{mn} = x_m \circ y_n\}, \end{aligned}$$

*and a set of $\binom{mn}{2} + \binom{m}{2}\binom{n}{2}$ linearly independent quadratic relations $\Re$ described below.*

**12.** $\Re = \Re_a \cup \Re_b$ is a disjoint union.

$\Re_a$ is the set of defining relations of the YB-algebra $\mathfrak{A} = \mathcal{A}(K, X \circ Y, \mathbf{r}_{X \circ Y})$ of the Cartesian prod. $(X \circ Y, r_{X \circ Y})$.
$\Re_a = \Re_{a1} \cup \Re_{a2}$ is a disjoint union of order $|\Re_a| = \binom{mn}{2}$.

$$\begin{aligned}
\Re_{a1} = \ & \{f_{ji,ba} = (x_j \circ y_b)(x_i \circ y_a) - (x_{i'} \circ y_{a'})(x_{j'} \circ y_{b'}), \\
& 1 \leq i, j \leq m, 1 \leq a, b \leq n, \text{ where} \\
& r_1(x_j x_i) = x_{i'} x_{j'}, \text{ with } j > i', \text{ and } r_2(y_b y_a) = y_{a'} y_{b'}\} \\
& LM(f_{ji,ba}) = (x_j \circ y_b)(x_i \circ y_a). \ |\Re_{a1}| = \tbinom{m}{2} n^2. \\
\Re_{a2} = \ & \{f_{ij,ba} = (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}), \\
& 1 \leq i, j \leq m, \ 1 \leq a, b \leq n, \text{ where} \\
& x_i x_j = r_1(x_i x_j) \text{ and } \ r_2(y_b y_a) = y_{a'} y_{b'}, \text{ with } b > a'\}. \\
& LM(f_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a). \ |\Re_{a2}| = m\tbinom{n}{2}.
\end{aligned}$$

$\Re_b$ consists of $\binom{m}{2}\binom{n}{2}$ relations:

$$\begin{aligned}
\Re_b = \ & \{g_{ij,ba} = (x_i \circ y_b)(x_j \circ y_a) - (x_i \circ y_{a'})(x_j \circ y_{b'}), \\
& 1 \leq i, j \leq m, 1 \leq a, b \leq n, \text{ where} \\
& r_1(x_i x_j) > x_i x_j, \ r_2(y_b y_a) = y_{a'} y_{b'} \text{ and } \ b > a'\}. \\
& LM(g_{ij,ba}) = (x_i \circ y_b)(x_j \circ y_a).
\end{aligned}$$

## Skip* The proof is in four steps.

(1) $\Re = \Re_a \cup \Re_b$ is contained in the ideal of relations $I = (\Re(A \circ B))$.

(2) We count

$$|\Re_a| = \binom{mn}{2}; \ |\Re_b| = \binom{m}{2}\binom{n}{2}; \ |\Re| = \binom{mn}{2} + \binom{m}{2}\binom{n}{2}.$$

(3) The set of polynomials $\Re \subset K\langle W \rangle$ is linearly independent.

(4)

$$I_2 \oplus (A \circ B)_2 = (K\langle W \rangle)_2.$$

$$\dim_K I_2 + \dim_K (A \circ B)_2 = m^2 n^2 = \dim_K (K\langle W \rangle)_2$$

$$\dim_K (Span \Re) + \dim_K (A \circ B)_2 = m^2 n^2$$

Hence $(\Re)_2 = I_2$, and $\Re$ generates the ideal of relations of the algebra $A \circ B$.

# 13. Segre maps of Yang-Baxter algebras

**Problem 2.** Introduce noncommutative analogues of *Segre maps in the class of Yang-Baxter algebras of finite solutions*. We have to find a solution $(Z, r_Z)$, with YB-algebra $\mathfrak{A}_Z$ and a homomorphism of graded algebras:

$$S : \mathfrak{A}_Z \longrightarrow A \otimes B,$$

s.t. $ImS = A \circ B$ and to find $\ker S$.

**Definition-Notation.** Let $Z = \{z_{11}, z_{12}, \cdots, z_{mn}\}$ be a set of order $mn$, disjoint with $X$ and $Y$. Define a map

$$\mathfrak{r} : Z \times Z \longrightarrow Z \times Z$$

induced canonically from the solution $(X \circ Y, r_{X \circ Y})$:

$$\begin{aligned} \mathfrak{r}(z_{jb}, z_{ia}) &= (z_{i'a'}, z_{j'b'}) \textit{ iff} \\ & r_{X\circ Y}(x_j \circ y_b, x_i \circ y_a) = (x_{i'} \circ y_{a'}, x_{j'} \circ y_{b'}). \end{aligned}$$

$(Z, \mathfrak{r})$ is a solution of YBE isomorphic to $(X \circ Y, r_{X \circ Y})$ (and to the Cartesian product $(X \times Y, r_{X \times Y})$).

Fix the deg-lex order on the free monoid $\langle Z \rangle$ induced by the enumeration

# 14. Segre maps of Yang-Baxter algebras

**Lemma**. In notation as above. Let $(X, r_1)$ and $(Y, r_2)$ be solutions on the finite disjoint sets $X = \{x_1, \cdots, x_m\}$, and $Y = \{y_1, \cdots, y_n\}$, and let $A = \mathcal{A}(K, X, r_1)$, and $B = \mathcal{A}(K, Y, r_2)$ be the corresponding YB algebras. Let $(Z, \mathfrak{r})$ be the solution of order $mn$ defined in Def-Notation, and let $\mathfrak{A}_Z = \mathcal{A}(K, Z, \mathfrak{r})$ be its YB algebra. Then the assignment

$$z_{11} \mapsto x_1 \otimes y_1,\ z_{12} \mapsto x_1 \otimes y_2,\ \cdots,\ z_{mn} \mapsto x_m \otimes y_n$$

extents to an algebra homomorphism

$$s_{m,n} : \mathfrak{A}_Z \longrightarrow A \otimes_K B.$$

**Definition.** We call the map $s_{m,n} : \mathfrak{A}_Z \longrightarrow A \otimes_K B$ *the $(m, n)$-Segre map*.

# 15. Assumptions and notations as above

$(X, r_1)$ and $(Y, r_2)$ are disjoint solutions

$$X = \{x_1, \cdots, x_m\}, \quad Y = \{y_1, \cdots, y_n\},$$

$A$ and $B$ - the corresponding Yang-Baxter algebras. $(Z, \mathfrak{r})$ is the solution on the set

$$Z = \{z_{11}, \cdots, z_{mn}\}$$

isom. to $(X \circ Y, r_{X \circ Y})$ and to the Cartesian product $(X \times Y, r_{X \times Y})$. $\mathfrak{A}_Z = \mathcal{A}(K, Z, \mathfrak{r})$ is its YB- algebra.
$s_{m,n} : \mathfrak{A}_Z \longrightarrow A \otimes_{\mathbf{k}} B$ is the Segre map extending the assignment

$$z_{11} \mapsto x_1 \circ y_1, \; z_{12} \mapsto x_1 \circ y_2, \; \cdots, \; z_{mn} \mapsto x_m \circ y_n.$$

**Theorem B.**

(1) *The image of the Segre map $s_{m,n}$ is the Segre product $A \circ B$. Moreover, $s_{m,n} : \mathfrak{A}_Z \longrightarrow A \circ B$ is a homomorphism of graded algebras.*

(2) *The kernel $\mathfrak{K} = \ker(s_{m,n})$ of the Segre map is generated by the set $\Re_s$ of $\binom{m}{2}\binom{n}{2}$ linearly independent quadratic binomials described below*

$$\begin{aligned}\Re_s = \quad & \{h_{ij,ba} = z_{ib}z_{ja} - z_{ia'}z_{jb'},\ 1 \leq i,j \leq m, 1 \leq a,b \leq n \mid \\ & r_1(x_ix_j) > x_ix_j, \text{ and } \ r_2(y_by_a) = y_{a'}y_{b'} \text{ with } b > a'\}.\end{aligned}$$

**Sketch of proof.** (i) $\Re_s$ consists of nonzero elements of $\mathfrak{A}_Z$.
(ii) $s_{m,n}(\Re_s) = \Re_b$ therefore $\Re_s \subset \mathfrak{K} = \ker(s_{m,n})$, moreover $\Re_s$ is linearly indept.
(iii) $\Re_s$ is a minimal set of generators of the kernel $\mathfrak{K}$.

# 17. Corollary.

Let $A = \mathcal{A}(K, X, r_1)$, and $B = \mathcal{A}(K, Y, r_2)$, be the Yang-Baxter algebras of the finite solutions $(X, r_1)$ and $(Y, r_2)$. Then the Segre product $A \circ B$ is a left and a right Noetherian algebra. Moreover, $A \circ B$ has polynomial growth. Moreover, $A \circ B$ is Koszul.

## 18. Open Question

(1) Let $A = \mathcal{A}(K, X, r_1)$, and $B = \mathcal{A}(K, Y, r_2)$, be the Yang-Baxter algebras of the finite solutions $(X, r_1)$ and $(Y, r_2)$. Is it true that the Segre product $A \circ B$ is a domain?

(2) Let $A = \mathcal{A}(K, X, r_1)$, and $B = \mathcal{A}(K, Y, r_2)$, be the YB algebras of the finite *square-free solutions* $(X, r_1)$ and $(Y, r_2)$. Is it true that the Segre product $A \circ B$ is a domain?

(3) Let $A$ and $B$ be binomial skew polynomial algebras. Is it true that the Segre product $A \circ B$ is a domain?

(2) and (3) are equivalent. We expect that due to the good algebraic and combinatorial properties of $A$ and $B$, the answer is affirmative. In cases (2) and (3) the Segre product $A \circ B$ is a PBW algebra whose quadratic relations are explicitly given. Observe that $A$ and $B$ are Noetherian domains, and $A \circ B$ is a subalgebra of the tensor product $A \otimes B$. However, it is shown by Rowen that the tensor product $D_1 \otimes_F D_2$ of two division algebras over an algebraically closed field contained in their centers may not be a domain.

# 19. Segre products and Segre maps for the YB algebras of square-free solutions

Among all Yang-Baxter algebras of finite solutions $A = \mathcal{A}(X, r)$ the only PBW algebras $A = \mathcal{A}(K, X, r)$ are those corresponding to square-free solutions.

**Theorem**

*(GI 2022) If $(X, r)$ is a finite solution of YBE then its Yang-Baxter algebra $\mathcal{A} = \mathcal{A}(K, X, r)$ is a PBW algebra with respect to a proper enumeration $X = \{x_1, x_2, \cdots, x_n\}$ iff $(X, r)$ is a square-free solution.*

**20.** From now on $(X, r_1)$ and $(Y, r_2)$ are disjoint square-free solutions,

$$X = \{x_1, \cdots, x_m\}, \quad \text{and} \quad Y = \{y_1, \cdots, y_n\}$$

are enumerated so that the Yang-Baxter algebras $A = \mathcal{A}(K, X, r_1)$, and $B = \mathcal{A}(K, Y, r_2)$ are *binomial skew polynomial rings* with respect to these enumerations.

## Theorem C.

*The Segre product $A \circ B$ satisfies the following conditions.*

(1) *$A \circ B$ is a PBW algebra with a set of $mn$ PBW generators*

$$W = X \circ Y = \begin{array}{l} \{w_{11} = x_1 \circ y_1,\ w_{12} = x_1 \circ x_2, \cdots, \\ \cdots, w_{1n} = x_1 \circ y_n, \cdots,\ w_{mn} = x_m \circ x_n\} \end{array}$$

*ordered lexicographically, and a standard finite presentation*

$$A \circ B \simeq K\langle w_{11}, \cdots, w_{mn}\rangle / (\Re),$$

*where the set of relations $\Re$ is a Gröbner basis of the ideal $I = (\Re)$ and consists of $\binom{mn}{2} + \binom{m}{2}\binom{n}{2}$ square-free quadratic polynomials described in Theorem A.*

(2) *$A \circ B$ is a Koszul algebra.*

(3) *$A \circ B$ is left and right Noetherian.*

(4) *The algebra $A \circ B$ has polynomial growth and* ***infinite global dimension****.*

## 22. Segre morphisms for YB algebras of square-free solutions

Theorem D below shows that our (noncommutative) analogue of Segre morphisms for Yang-Baxter algebras of finite solutions (the general case) can be defined also for the subclass of Yang-Baxter algebras related to square-free solutions. This is in contrast with our recent results on Veronese subalgebras which imply that the noncommutative analogue of Veronese morphisms for the class of Yang-Baxter algebras related to (arbitrary) finite solutions of YBE, introduced in [GI22] can not be restricted to the subclass of YB algebras of square-free solutions.

**Hypothesis of Theorem D.** Assumptions and notation as above. Suppose $(X, r_1)$ and $(Y, r_2)$ are disjoint square-free solutions, $X = \{x_1, \cdots, x_m\}$, $Y = \{y_1, \cdots, y_n\}$ enumerated so that the Yang-Baxter algebras $A = \mathcal{A}(K, X, r_1)$, and $B = \mathcal{A}(K, Y, r_2)$ are *binomial skew polynomial rings* w.r.t. these enumerations. Let $(Z, r_Z)$ be *the square-free solution* on the set $Z = \{z_{11}, \cdots, z_{mn}\}$, isomorphic the Cartesian product of

## Theorem D.

*Let* $(Z, r_Z)$ *be the square-free solution on the set* $Z = \{z_{11}, \cdots, z_{mn}\}$, *isomorphic the Cartesian product of solutions* $(X \circ Y, \mathfrak{r})$, *and let* $\mathfrak{A} = \mathcal{A}(K, Z, r_Z)$ *be its YB algebra. (We know that* $\mathfrak{A}$ *is also a binomial skew-polynomial ring). Let*

$$s_{m,n} : \mathfrak{A} = \longrightarrow A \otimes_{\boldsymbol{k}} B$$

*be the Segre map extending the assignment*
$z_{11} \mapsto x_1 \circ y_1, \;\; z_{12} \mapsto x_1 \circ y_2, \cdots, z_{mn} \mapsto x_m \circ y_n.$

(1) *The image of the Segre map* $s_{m,n}$ *is the Segre product* $A \circ B$.

(2) *The kernel* $\mathfrak{K} = \ker(s_{m,n})$ *is generated by the set of* $\binom{m}{2}\binom{n}{2}$ *linearly independent quadratic binomials listed below:*

$$h_{ij,ba} = z_{ib}z_{ja} - z_{ia'}z_{jb'}, 1 \leq i < j \leq m, 1 \leq a < b \leq n$$
$$\text{where } r_2(y_b y_a) = y_{a'} y_{b'} \text{ with } \; b > a', a' < b'.$$

# 24. An Example of $A \circ B$

$$
\begin{aligned}
A &= \mathcal{A}(K, X, r_1) = K\langle x_1, x_2, x_3\rangle / (x_3x_2 - x_1x_3, x_3x_1 - x_2x_3, x_2x_1 - x_1x_2) \\
B &= \mathcal{A}(K, Y, r_2) = K\langle y_1, y_2\rangle / (y_2^2 - y_1^2).
\end{aligned}
$$

$A$ is a binomial skew-polynomial ring, its rel. form a Gröbner basis of the ideal they generate. The relations of $B$ do not form a Gröbner basis of the ideal $J = (y_2^2 - y_1^2)$. The reduced Gr. basis of $J$ is $G = \{y_2^2 - y_1^2,\ y_2y_1y_1 - y_1y_1y_2\}$.
Let $A \circ B$ be the Segre product of $A$ and $B$, and let $(X \circ Y, r_{X \circ Y})$ be the solution isomorphic to the Cartesian product of solutions $(X \times Y, \mathfrak{r})$.
$A \circ B$ is a quadratic algebra with a set of **6** one-generators

$$
\begin{aligned}
W = \{ \quad & w_{11} = x_1 \circ y_1, w_{12} = x_1 \circ y_2, w_{21} = x_2 \circ y_1, \\
& w_{22} = x_2 \circ y_2, w_{31} = x_3 \circ y_1, w_{32} = x_3 \circ y_2\}
\end{aligned}
$$

and **18** defining quadratic relations.

$$A \circ B \simeq K\langle w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\rangle / (\mathfrak{R})$$

$\mathfrak{R} = \mathfrak{R}_a \cup \mathfrak{R}_b$ is a disjoint union of quadratic relations, where $\mathfrak{R}_a$ are the relations of the YB algebra $\mathfrak{A}_{X \circ Y}$ with $|\mathfrak{R}_a| = 15$, $\mathfrak{R}_a = \mathfrak{R}_{a1} \cup \mathfrak{R}_{a2}$,

$$\begin{aligned}
\mathfrak{R}_{a1} = \{ \quad & f_{32,22} = w_{32}w_{22} - w_{11}w_{31}, \quad & f_{32,11} = w_{31}w_{21} - w_{12}w_{32}, \\
& f_{32,21} = w_{32}w_{21} - w_{12}w_{31}, \quad & f_{32,12} = w_{31}w_{22} - w_{11}w_{32}, \\
& f_{31,22} = w_{32}w_{12} - w_{21}w_{31}, \quad & f_{31,11} = w_{31}w_{11} - w_{22}w_{32}, \\
& f_{31,21} = w_{32}w_{11} - w_{22}w_{31}, \quad & f_{31,12} = w_{31}w_{12} - w_{21}w_{32}, \\
& f_{21,22} = w_{22}w_{12} - w_{11}w_{21}, \quad & f_{21,11} = w_{21}w_{11} - w_{12}w_{22}, \\
& f_{21,21} = w_{22}w_{11} - w_{12}w_{21}, \quad & f_{21,12} = w_{21}w_{12} - w_{11}w_{22} \quad \}.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{R}_{a2} = \{ \quad & f_{33,22} = w_{32}w_{32} - w_{31}w_{31}, \quad & f_{22,22} = w_{22}w_{22} - w_{21}w_{21}, \\
& f_{11,22} = w_{12}w_{12} - w_{11}w_{11} \ \}.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{R}_b = \{ \quad & g_{23,22} = w_{22}w_{32} - w_{21}w_{31}, \ g_{13,22} = w_{12}w_{32} - w_{11}w_{31}, \\
& g_{12,22} = w_{12}w_{22} - w_{11}w_{21} \}.
\end{aligned}$$

Let $(Z, r_Z)$ be the solution isomorphic to the Cartesian product $(X \circ Y, r_{X \circ Y})$, where

$Z = \{z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}\}$. The YB algebra $\mathfrak{A}_Z = \mathcal{A}(K, Z, r_Z)$ has a finite presentation

$$\mathfrak{A}_Z = K\langle z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}\rangle / (\Re(\mathfrak{A}_Z)),$$

$$\begin{array}{rll}\Re(\mathfrak{A}_Z) = \{ & f_{32,22} = z_{32}z_{22} - z_{11}z_{31}, & f_{32,11} = z_{31}z_{21} - z_{12}z_{32}, \\ & f_{32,21} = z_{32}z_{21} - z_{12}z_{31}, & f_{32,12} = z_{31}z_{22} - z_{11}z_{32}, \\ & f_{31,22} = z_{32}z_{12} - z_{21}z_{31}, & f_{31,11} = z_{31}z_{11} - z_{22}z_{32}, \\ & f_{31,21} = z_{32}z_{11} - z_{22}z_{31}, & f_{31,12} = z_{31}z_{12} - z_{21}z_{32}, \\ & f_{21,22} = z_{22}z_{12} - z_{11}z_{21}, & f_{21,11} = z_{21}z_{11} - z_{12}z_{22}, \\ & f_{21,21} = z_{22}z_{11} - z_{12}z_{21}, & f_{21,12} = z_{21}z_{12} - z_{11}z_{22}, \\ & f_{33,22} = z_{32}z_{32} - z_{31}z_{31}, & f_{22,22} = z_{22}z_{22} - z_{21}z_{21}, \\ & f_{11,22} = z_{12}z_{12} - z_{11}z_{11} \ \}. & \end{array}$$

$(Z, r_Z)$ is not a square-free solution, and therefore, the defining relations $\Re(\mathfrak{A}_Z)$ do not form a Gröbner basis.

The Segre map $s_{3,2} : \mathfrak{A}_Z \longrightarrow A \otimes B$ has image $A \circ B$. The kernel $\ker(s_{3,2})$ is the ideal of $\mathfrak{A}_Z$ generated by the following three polynomials

$$
\begin{aligned}
t_{23,22} &= z_{22}z_{32} - z_{21}z_{31}, \\
t_{13,22} &= z_{12}z_{32} - z_{11}z_{31}, \\
t_{12,22} &= z_{12}z_{22} - z_{11}z_{21}.
\end{aligned}
$$