On ring-theoretical properties of Yang–Baxter algebras

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Groups, rings and the Yang–Baxter equation
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The YBE originated (in 60's and 70's) in the papers of Yang and Baxter devoted to the many-body problem in statistical mechanics.

Nowadays, the YBE is considered as one of the most fundamental equations in mathematical physics. It also laid foundations for several important areas of mathematics, including theory of quantum groups and Hopf algebras.

Definition

A (set-theoretic) solution of the YBE is a pair (X, r), where X is a non-empty set and $r\colon X^2 \to X^2$ is an arbitrary map satisfying

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where $r_{12} = r \times \mathrm{id}$ and $r_{23} = \mathrm{id} \times r$ are maps from X^3 to itself.

For a solution (X, r) of the YBE we write

$$r(x, y) = (\lambda_x(y), \rho_y(x)).$$

Moreover, we say that the solution (X, r) is:

- finite if $|X| < \infty$.
- bijective if r is a bijective map.
- involutive if $r^2 = \mathrm{id}$.
- idempotent if $r^2 = r$.
- left (respectively right) non-degenerate if all the maps λ_x (respectively ρ_y) are bijective. If (X, r) is left and right non-degenerate then we simply call it a non-degenerate solution.

Main problem (Drinfeld, 1990)

Construct and classify (up to isomorphism) all set-theoretic solutions of the YBE.

Definition

The structure monoid of a solution (X, r) of the YBE is defined by the following presentation

$$M(X,r) = \langle X \mid xy = uv \text{ if } r(x,y) = (u,v) \rangle$$
$$= \langle X \mid xy = \lambda_x(y)\rho_y(x) \text{ for all } x, y \in X \rangle.$$

If K is a field then the monoid algebra

$$K[M(X,r)]$$

is called the structure algebra of the solution (X, r).

Motivation. On the one hand these structures encode a lot of information about the solutions themselves. On the other hand they are natural examples of graded quadratic (and finitely presented provided X is finite) monoids and algebras with nice structural and homological properties.

Non-degenerate involutive solutions

A first exciting result on ring-theoretical properties of Yang–Baxter algebras goes back to work of Gateva-Ivanova and Van den Bergh on the so-called semigroups of I-type.

Theorem (Gateva-Ivanova, Van den Bergh, 1998)

Let (X, r) be a finite non-degenerate involutive solution of the YBE. If K is a field and $R = K[M(X, r)]$ then:

1. R is a Noetherian PI domain.
2. R is a maximal order in its quotient division ring.
3. R is Auslander-regular, Cohen–Macaulay and Koszul.

Moreover,

$$\operatorname{GKdim} R = \operatorname{clKdim} R = |X|.$$

Remark. Structure algebras of such solutions share many good properties with commutative polynomial algebras in finitely many variables.

1. Let $|X| > 1$ and $r(x, y) = (x, y)$ for $x, y \in X$. Then (X, r) is an involutive but left and right degenerate solution. Moreover, $K[M(X, r)] \cong K\langle X\rangle$ is the free algebra on X.

2. Let $X = \{1, 2, 3\}$ and $r(x, y) = (y, \sigma_y(x))$ for $x, y \in X$, where $\sigma_1 = (23)$, $\sigma_2 = (13)$ and $\sigma_3 = (12)$. Then (X, r) is a non-degenerate bijective solution (with $r^3 = \mathrm{id}$). Moreover,

$$K[M(X, r)] \cong K\langle x, y, z \mid xy = yz = zx \text{ and } zy = yx = xz\rangle$$

is Noetherian and PI, but not semiprime (let alone domain).

3. Let $X = \{1, 2\}$ and $r(x, y) = (y, y)$ for $x, y \in X$. Then (X, r) is an idempotent, left but not right non-degenerate solution. Moreover,

$$K[M(X, r)] \cong K\langle x, y \mid xy = y^2 \text{ and } yx = x^2\rangle$$

is left Noetherian and PI, but neither right Noetherian nor semiprime.

Conclusion. In order to obtain Yang–Baxter algebras with nice properties, one needs to impose some constraints on the solutions.

For example if $R = K[M(X, r)]$ is left or right Noetherian and I is the ideal of R generated by all elements of the form xy for $x, y \in X$ then

$$R/I = K + \bigoplus_{x \in X} K \cdot \overline{x}$$

is a commutative Noetherian local algebra. Moreover, the unique maximal ideal of R/I, generated by all the cosets $\overline{x} = x + I$ for $x \in X$, is of square zero and it is finitely generated iff X is a finite set.

Similarly, as degenerate solutions may lead to Yang–Baxter algebras containing non-commutative free subalgebras, we shall consider one-sided non-degenerate solutions.

Since now assume (X, r) is a finite left non-degenerate solution.

In tackling the problem of Noetherianity of the Yang–Baxter algebra $K[M(X, r)]$ the following result plays a fundamental role.

Theorem (Okniński, 2001)

Assume S is a finitely generated monoid with an ideal chain

$$\varnothing = S_{n+1} \subseteq S_n \subseteq \cdots \subseteq S_1 \subseteq S_0 = S$$

such that each factor S_i/S_{i+1} for $0 \leqslant i \leqslant n$ is either power nilpotent or a uniform subsemigroup of a Brandt semigroup (i.e., a completely 0-simple inverse semigroup). Let K be a field. If S satisfies the ascending chain condition on right ideals and $\operatorname{GKdim} K[S]$ is finite then $K[S]$ is right Noetherian.

Another important tool is the so-called left derived monoid of (X, r) given by the following presentation

$$A(X,r) = \langle X \mid x\lambda_x(y) = \lambda_x(y)\lambda_{\lambda_x(y)}(\rho_y(x)) \text{ for all } x, y \in X\rangle.$$

Note that actually

$$A(X,r) = M(X,s) = \langle X \mid xy = y\sigma_y(x) \text{ for all } x, y \in X\rangle,$$

where (X, s) is the so-called left derived solution of (X, r) defined as

$$s(x,y) = (y, \sigma_y(x)), \qquad \text{where} \qquad \sigma_y(x) = \lambda_y(\rho_{\lambda_x^{-1}(y)}(x)).$$

Remark. The monoid $A(X, r)$ encodes behaviour of the map r^2. In particular, if the solution (X, r) is involutive then $A(X, r)$ is the free abelian monoid on X.

Theorem (Colazzo, Jespers, Van Antwerpen, Verwimp, 2022)

Let (X, r) be a finite left non-degenerate solution of the YBE. If K is a field and $A = A(X, r)$ then $K[A]$ is a left Noetherian PI algebra with $\operatorname{GKdim} K[A] \leqslant |X|$.

Following the ideas of Soloviev, and Lu, Yan and Zhu one may prove that there exists an action of the monoid $M = M(X, r)$ on $A = A(X, r)$, i.e., a morphism of monoids $\lambda \colon M \to \operatorname{Aut}(A)$, and a bijective 1-cocycle $\pi \colon M \to A$ with respect to this action such that $\lambda(x) = \lambda_x$ and $\pi(x) = x$ for all $x \in X$.

Consequence. The monoid M may be identified with a regular submonoid of the semidirect product

$$M \cong \{(a, \lambda_a) : a \in A\} \subseteq A \rtimes \operatorname{Im} \lambda.$$

In particular,

$$\operatorname{GKdim} K[M] = \operatorname{GKdim} K[A] \leqslant |X| < \infty.$$

Moreover, as $K[A \rtimes \operatorname{Im}\lambda]$ is a finite left $K[A]$-module, the previous theorem imply that $K[A \rtimes \operatorname{Im}\lambda]$ is PI. Hence, its subalgebra $K[M]$ is PI as well.

Next, as $K[A]$ is left Noetherian, A has ACC on left ideals. Since

$$ab = b\sigma_b(a) \text{ for } a, b \in A \implies Ab \subseteq bA,$$

it follows that each right ideal of A is a two-sided ideal. Therefore, A has ACC on right ideals as well.

Furthermore, the rule

$$R \mapsto R^e = \{(a, \lambda_a) : a \in R\}$$

defines a bijection between the sets of right ideals of A and that of M. Hence, M also has ACC on right ideals.

So, to decide when $K[M]$ is right Noetherian it remains to construct an ideal chain in M with properties as in Okniński's theorem. This is the most technical part and it is achieved by considering left divisibility in M.

First, we have an ideal chain

$$\varnothing = M_{n+1} \subseteq M_n \subseteq \cdots \subseteq M_1 \subseteq M_0 = M,$$

where $n = |X|$ and M_i consists of elements in M that are left divisible by at least i generators from X.

Next, we search for necessary and sufficient conditions in order to refine this chain to an ideal chain making use of some intermediate ideals

$$M_{i+1} \subseteq B_i \subseteq U_i \subseteq M_i$$

and such that we have the following properties on the Rees factor semigroups B_i/M_{i+1}, U_i/B_i, M_i/U_i and M_i/M_{i+1}:

- B_i/M_{i+1} and M_i/U_i are power nilpotent semigroups (if M_i/M_{i+1} is power nilpotent then we take $B_i = U_i = M_i$).
- if M_i/M_{i+1} is not power nilpotent semigroup then U_i/B_i is a disjoint union of semigroups $S_1, \dots, S_m$ such that $S_k S_l \subseteq M_{i+1}$ for $k \neq l$.
- each $(S_i \cup M_{i+1})/M_{i+1}$ is a uniform subsemigroup of a Brand semigroup.

The intermediate ideals B_i and U_i are build with the use of the sets

$$M_{YZ} = \{(a, \lambda_a) \in M : a \text{ is left divisible precisely by generators from } Y \text{ and } \lambda_a(Z) = Y\}$$

for $Y, Z \subseteq X$ with $|Y| = |Z| = i$.

Finally, if $Y \subseteq X$ is such that M_{YY} is a (non-empty) semigroup then there exists an element $m_Y \in M_{YY}$ of the form $m_Y = (a_Y, \mathrm{id})$.

Now, we are ready to state one of the main results.

Theorem (CJKVA 2023 = Colazzo, Jespers, ŁK, Van Antwerpen, 2023)

Assume (X, r) is a finite left non-degenerate solution of the YBE. If K is a field and $M = M(X, r)$ then:

1. if for each subset $Y \subseteq X$ for which M_{YY} is a (non-empty) semigroup there exists $d \geqslant 1$ such that $m_Y^d M_{YY}$ is a cancellative semigroup then $K[M]$ is right Noetherian.
2. the cancellative assumption holds for the semigroup $m_Y^d M_{YY}$ iff for some $k \geqslant 1$ one has $a^k b^k = b^k a^k$ for all $a, b \in m_Y^d M_{YY}$.
3. if M also satisfies the ascending chain condition on left ideals (e.g., if (X, r) is additionally bijective) then $K[M]$ also is left Noetherian.

We further reduce the problem of studying right Noetherianity of $K[M]$ to $K[S]$ for an easier, but closely related monoid

$$S = \mathrm{Soc}(M) = \{(a, \lambda_a) \in M : \lambda_a = \mathrm{id}\},$$

called the socle of M, and to the sets $S_{YY} = M_{YY} \cap S$.

Theorem (CJKVA 2023)

Let (X, r) be a finite left non-degenerate solution of the YBE. If K is a field, $M = M(X, r)$ and $S = \mathrm{Soc}(M)$ then $K[M]$ is right Noetherian iff $K[S]$ is right Noetherian, which is further equivalent to the existence of $d \geqslant 1$ such that S_{YY}^d is a cancellative semigroup for each subset $Y \subseteq X$ for which the Rees factor semigroup $(S_{YY} \cup M_{|Y|+1})/M_{|Y|+1}$ is not nil.

Remark. Note that the socle $\mathrm{Soc}(M)$ may be treated as a submonoid of both M and $A = A(X, r)$.

Despite our theorems may look a bit technical (but conditions that appear are of finitary nature!), they have various consequences; some of them we proved before (in 2019 and 2022) by using completely different methods.

Corollary (CJKVA 2023)

Assume (X, r) is a finite left non-degenerate solution of the YBE. If K is a field, $A = A(X, r)$ and $M = M(X, r)$ then:

1. if $K[A]$ is right Noetherian then so is $K[M]$.
2. if A satisfies the left Ore condition (e.g., if A is Malcev nilpotent) then $K[M]$ is right Noetherian. Moreover, if A is abelian then $K[M]$ also is left Noetherian.
3. if $K[M]$ is right Noetherian or semiprime then M^d_{XX} is cancellative for some $d \geqslant 1$. In particular, M^d_{XX} has a group of fractions that is abelian-by-finite.
4. if (X, r) is idempotent then $K[M]$ is left Noetherian and $\operatorname{GKdim} K[M] = 1$. Furthermore, $K[M]$ is right Noetherian iff the set $\{\lambda_x^{-1}(x) : x \in X\}$ is a singleton iff for some $k \geqslant 1$ one has $a^k b^k = b^k a^k$ for all $a, b \in M$.

It turns out that the primeness of $K[M]$ it controlled by behaviour of a certain finite subgroup $\Omega(X,r)$ of the structure group $G(X,r)$, which is an orbit of a single element under the action of the permutation group $\langle \lambda_x : x \in X \rangle$.

Theorem (CJKVA 2023)

Let (X,r) be a finite left non-degenerate solution of the YBE. If K is a field, $M = M(X,r)$ and $\Omega = \Omega(X,r)$ then the following conditions are equivalent:

1. (X,r) is an involutive solution.
2. M is a cancellative monoid and Ω is a trivial group.
3. $K[M]$ is a prime algebra and Ω is a trivial group.
4. $K[M]$ is a domain.

Furthermore, if the diagonal map $x \mapsto \lambda_x^{-1}(x)$ is a permutation of X (e.g., if (X,r) is bijective) then Ω is a trivial group.

Remark. The equivalence of (1) and (2) for non-degenerate bijective solutions was conjectured by Gateva-Ivanova.

We characterise when the algebra $K[A]$ is semiprime in terms of a decomposition of the monoid A as a finite semilattice of certain cancellative subsemigroups.

Theorem (CJKVA 2023)

Let (X, r) be a finite non-degenerate bijective solution of the YBE. If K is a field and $A = A(X, r)$ then the following conditions are equivalent:

1. $K[A]$ is a semiprime algebra.
2. A is a disjoint union $A = \bigcup_{e \in \Gamma} A_e$ of cancellative semigroups A_e indexed by a finite semilattice Γ such that $A_e A_f \subseteq A_{ef}$ for all $e, f \in \Gamma$ (i.e., A is a finite semilattice Γ of cancellative semigroups) and each $K[A_e]$ is semiprime.

Moreover, in case the above equivalent conditions hold, Γ is the set of central idempotents of the classical ring of quotients of $K[A]$. Equivalently, A has a finite ideal chain with Rees factors cancellative semigroups that yield semiprime semigroup algebras. The latter condition holds in case K has zero characteristic.

Remark. If $Q = Q_{cl}(K[A])$ then $A_e = \{a \in A : Qa = Qe\}$ for $e \in \Gamma$.

Further, a description of prime ideals of $K[A]$ leads to a purely combinatorial formula for the Gelfand–Kirillov dimension of $K[M]$.

Theorem (CJKVA 2023)

Let (X,r) be a finite left non-degenerate solution of the YBE. If K is a field, $M=M(X,r)$ and $A=A(X,r)$ then

$$\operatorname{GKdim} K[M] = \operatorname{GKdim} K[A] = \operatorname{clKdim} K[A].$$

Moreover, if

$$\mathcal{Y} = \{\varnothing \neq Y \subseteq X : \sigma_y(Y) \subseteq Y \text{ and } \sigma_y(X \setminus Y) \subseteq X \setminus Y \text{ for all } y \in Y\}$$

then the above (equal) dimensions are also equal to the maximum of numbers n_Y for $Y \in \mathcal{Y}$, where n_Y is the number of orbits of the set Y with respect to the action of the monoid $\Sigma_Y = \langle \sigma_y : y \in Y \rangle$. In particular, all the above dimensions are bounded by $|X|$. Furthermore, if $K[M]$ is left or right Noetherian then also

$$\operatorname{GKdim} K[M] = \operatorname{clKdim} K[M].$$

As a consequence of previous theorems we obtain the following (homological) characterisation of involutive solutions.

Theorem (CJKVA 2023)

Let (X, r) be a finite left non-degenerate solution of the YBE. If K is a field and $M = M(X, r)$ then the following conditions are equivalent:

1. (X, r) is an involutive solution.
2. $\operatorname{GKdim} K[M] = |X|$.

Moreover, if $K[M]$ is left and right Noetherian then the above conditions are equivalent to:

3. $\operatorname{rk} M = |X|$.
4. $\operatorname{clKdim} K[M] = |X|$.
5. $\operatorname{id} K[M] = |X|$.
6. $K[M]$ has finite global dimension.
7. $K[M]$ is Auslander–Gorenstein with $\operatorname{id} K[M] = |X|$.
8. $K[M]$ is Auslander-regular.

Some open problems

Assume (X, r) is a finite left non-degenerate solution of the YBE. Let K be a field and $M = M(X, r)$.

1. When is $K[M]$ left Noetherian?
2. Do there exist prime algebras $K[M]$ with $\Omega(X, r)$ non-trivial?
3. When is $K[M]$ semiprime? If (X, r) is additionally bijective with $K[M]$ semiprime then there exist finitely many finitely generated abelian-by-finite groups $G_1, \ldots, G_k$, each being the group of quotients of a cancellative subsemigroup of M, such that $K[M]$ embeds into the direct product of matrix algebras $\mathrm{M}_{n_1}(K[G_1]) \times \cdots \times \mathrm{M}_{n_k}(K[G_k])$.
4. What about degenerate solutions? For example when Yang–Baxter algebras of such solutions are left or right Noetherian?

A first result on Noetherianity for certain degenerate solutions

Theorem (CJKVA 2023)

Let (X, r) be a finite solution of the YBE of the form $r(x, y) = (\lambda(y), \rho_y(x))$ (i.e., $\lambda_x = \lambda$ for all $x \in X$). If K is a field then $K[M(X, r)]$ is a left Noetherian PI algebra of finite Gelfand–Kirillov dimension. If, furthermore, the solution (X, r) is right non-degenerate then $K[M(X, r)]$ also is right Noetherian.

Remark. Put

$$\eta_i = \operatorname{Ker} \lambda^i = \{(x, y) \in X^2 : \lambda^i(x) = \lambda^i(y)\}.$$

Then $\eta_n = \eta_{n+1}$ for some $n \geqslant 1$. Defining the retraction relation $\sim$ on X by

$$x \sim y \iff \lambda^n(x) = \lambda^n(y) \text{ and } \rho_x = \rho_y$$

it is easy to check that (X, r) induces an irretractable solution $(\overline{X}, \overline{r})$ on the set $\overline{X} = X/\sim$. Moreover, $K[M(X, r)]$ is right Noetherian iff $K[M(\overline{X}, \overline{r})]$ is so.

Details can be found in:

I. Colazzo, E. Jespers, Ł. Kubat and A. Van Antwerpen
Structure algebras of finite set-theoretic solutions of the Yang–Baxter equation, arXiv:2305.06023 (2023).

Thank you
for your attention!