Strong semilattices of skew braces

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Joint work with Francesco Catino and Paola Stefanelli

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Solutions of the Yang-Baxter equation

If S is a set, a map $r : S \times S \longrightarrow S \times S$ satisfying the braid relation

$$(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)(r \times \mathrm{id}_S) = (\mathrm{id}_S \times r)(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)$$

is called *set-theoretic solution*, or briefly *solution*, of the Yang-Baxter equation.

For a solution r, we introduce two maps $\lambda_a, \rho_b : S \to S$ and write

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

for all $a, b \in S$. In particular, the solution r is said to be

- *left non-degenerate* if λ_a is bijective, for every $a \in S$;
- *right non-degenerate* if ρ_b is bijective, for every $b \in S$;
- *non-degenerate* if r is both left and right non-degenerate.

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Solutions associated to skew braces

[Rump, 2007] traced a novel research direction for finding solutions by introducing the algebraic structure of *brace*. Interesting generalizations have been produced over the years.

[Rump, 2007 - Guarnieri, Vendramin, 2017]

A triple $(B, +, \circ)$ is called *skew brace* if $(B, +)$ and $(B, \circ)$ are groups and it holds

$$\forall a, b, c \in B \qquad a \circ (b + c) = a \circ b - a + a \circ c.$$

If $(B, +)$ is abelian, then $(B, +, \circ)$ is a *brace*.

Any skew brace B gives rise to a *non-degenerate bijective solution*

$$r_B(a, b) = \left(-a + a \circ b, (-a + a \circ b)^{-} \circ a \circ b\right)$$

that is *involutive*, i.e., $r^2 = \mathrm{id}_{B\times B}$, if and only if $(B, +, \circ)$ is a brace.

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The opposite skew brace

If $(B, +, \circ)$ is a skew brace, one can consider the skew brace

$$B^{op} = (B, +^{op}, \circ)$$

with $a +^{op} b = b + a$, called the *opposite skew brace* of $(B, +, \circ)$.

As shown by **[Koch, Truman, 2020]**, considered the solution $r_{B^{op}}$ associated to the skew brace B^{op}

$$r_{B^{op}}(a, b) = \left(a \circ b - a, (a \circ b - a)^{-} \circ a \circ b\right),$$

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Semi-braces

[Catino, Colazzo, Stefanelli, 2017]

A *(left cancellative) semi-brace* is a triple $(S, +, \circ)$ such that $(S, +)$ is a left cancellative semigroup, $(S, \circ)$ is a group, and

$$\forall a, b, c \in S \qquad a \circ (b + c) = a \circ b + a \circ (a^- + c),$$

where a^- denotes the inverse of a with respect to $\circ$.

Every skew brace $(B, +, \circ)$ is a (left cancellative) semi-brace since

$$a \circ (a^- + c) = -a + a \circ c,$$

for all $a, c \in B$.

If $(S, +, \circ)$ is a (left cancellative) semi-brace, the map

$$r_S(a, b) = \left(a \circ (a^- + b), (a^- + b)^- \circ b\right)$$

is a *left non-degenerate solution.*

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is a solution if and only if

$$\forall a, b, x \in S \qquad x + a \circ (0 + b) = a + \lambda_a(b) \circ (0 + \rho_b(a)),$$

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Inverse semi-braces

[Catino, M., Stefanelli, 2021]

An *inverse semi-brace* is a triple $(S, +, \circ)$ such that $(S, +)$ is a semigroup, $(S, \circ)$ is an inverse semigroup, and

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where a^{-} denotes the inverse of a with respect to the $\circ$.

A semigroup $(S, \circ)$ is called *inverse* if, for each $a \in S$, there exists a unique $a^{-} \in S$ satisfying

$$a \circ a^{-} \circ a = a \quad \text{and} \quad a^{-} \circ a \circ a^{-} = a^{-}.$$

Every group is an inverse semigroup and the idempotent elements commute each other.

Under suitable conditions, the map r_S associated to an inverse semi-brace $(S, +, \circ)$ is a solution.

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Weak braces

Definition (Catino, M., Miccoli, Stefanelli, 2022)

A *weak brace* is a triple $(S, +, \circ)$ such that $(S, +)$ and $(S, \circ)$ are inverse semigroups satisfying

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where $-a$ and purple a^{-} denote the inverses of $(S, +)$ and $(S, \circ)$.

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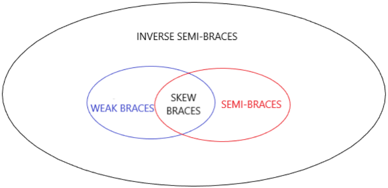

Idempotents of weak braces

In any weak brace $\mathrm{E}(S,+) = \mathrm{E}(S,\circ)$ thus we will simply write $\mathrm{E}(S)$.
As a consequence, if $|\mathrm{E}(S)| = 1$, then S is a skew brace.

Key Lemma

Let $(S,+,\circ)$ be a weak brace. Then, it holds

$$e + a = e \circ a,$$

for all $e \in \mathrm{E}(S)$ and $a \in S$.

In particular, $\mathrm{E}(S)$ also is a (trivial) weak brace.

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Given a weak brace $(S, +, \circ)$, we can consider its *opposite weak brace* that is $S^{op} = (S, +^{op}, \circ)$, with $a +^{op} b = b + a$, for all $a, b \in S$.

The solution $r_{S^{op}}$ associated to the opposite weak brace S^{op} of S is such that

$$r_S\, r_{S^{op}}\, r_S = r_S, \qquad r_{S^{op}}\, r_S\, r_{S^{op}} = r_{S^{op}}, \qquad \text{and} \qquad r_S r_{S^{op}} = r_{S^{op}} r_S.$$

Hence, r_S is a completely regular element of $\mathrm{Map}(S \times S)$.

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Structural properties of weak braces

Theorem

Let $(S, +, \circ)$ be a weak brace. Then, $(S, +)$ is a Clifford semigroup.

An inverse semigroup S is a *Clifford semigroup* if it has central idempotents.

Generally, $(S, \circ)$ is not a Clifford semigroup.

Example

Let $X := \{1, x, y\}$, S the upper semilattice on X with join 1, and T the commutative inverse monoid on X with identity 1 such that $xx = yy = x$ and $xy = y$. Consider the trivial weak braces on S and T. Let $\tau := (xy) \in \mathrm{Aut}(S)$ and $\sigma : T \to \mathrm{Aut}(S)$ defined by $\sigma(1) = \sigma(x) = \mathrm{id}_S$ and $\sigma(y) = \tau$. Then, $S \rtimes_\sigma T$ is a weak brace such that $(S \times T, \circ)$ is not Clifford, since

$$(y, y) \circ (y, y)^- = (y, x) \quad \& \quad (y, y)^- \circ (y, y) = (x, x).$$

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Weak braces coming from RB-operators

Examples of weak brace having can be obtained by using Rota-Baxter operators:

Definition (Catino, M., Stefanelli, 2023)

If $(S,+)$ is a Clifford semigroup, any map $\mathfrak{R}: S \to S$ satisfying

$$\forall a, b \in S \qquad \begin{aligned} \mathfrak{R}(a) + \mathfrak{R}(b) &= \mathfrak{R}(a + \mathfrak{R}(a) + b - \mathfrak{R}(a)) \\ a + \mathfrak{R}(a) - \mathfrak{R}(a) &= a \end{aligned}$$

is called *Rota–Baxter operator* on $(S,+)$.

Let $\mathfrak{R}$ an RB-operator on a Clifford semigroup $(S,+)$. Set

$$\forall a, b \in S \qquad a \circ_{\mathfrak{R}} b := a + \mathfrak{R}(a) + b - \mathfrak{R}(a),$$

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Dual weak braces

Definition

A weak brace $(S, +, \circ)$ is called dual weak brace if $(S, \circ)$ is a Clifford semigroup.

If $(S, +, \circ)$ is a dual weak brace, the solution r_S has also a *behaviour close to the non-degeneracy* in the sense that

$$\lambda_a \lambda_{a^-} \lambda_a = \lambda_a, \qquad \lambda_{a^-} \lambda_a \lambda_{a^-} = \lambda_{a^-}, \quad \text{and} \quad \lambda_a \lambda_{a^-} = \lambda_{a^-} \lambda_a$$

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for every $a \in S$.

Strong semilattice of skew braces

Let us consider the following:

- Y a (lower) semilattice;
- $\{B_\alpha \mid \alpha \in Y\}$ a family of disjoint skew braces;
- For each pair α, β of elements of Y such that $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ be a homomorphism of skew braces such that
  1. $\phi_{\alpha,\alpha}$ is the identical automorphism of B_α, for every $\alpha \in Y$;
  2. $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ if $\alpha \geq \beta \geq \gamma$.

Then, $S = \bigcup\{B_\alpha \mid \alpha \in Y\}$ endowed with the operation given by

$$\forall a \in B_\alpha, b \in B_\beta \quad a + b := \phi_{\alpha,\alpha\beta}(a) + \phi_{\beta,\alpha\beta}(b)$$
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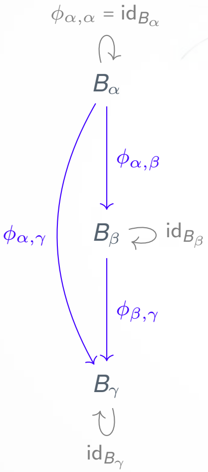

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Theorem (Catino, M., Stefanelli, 2023)

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Conversely, any dual weak brace is a strong semilattice of skew braces.

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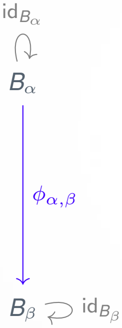
An easy example



Let us consider:

- $Y = \{\alpha, \beta\}$, with $\alpha > \beta$,
- B_α the trivial skew brace on the cyclic group C_3,
- B_β the trivial skew brace on the symmetric group Sym_3
- $\phi_{\alpha,\beta} : C_3 \to \mathrm{Sym}_3$ the homomorphism given by $\phi_{\alpha,\beta}(0) = \mathrm{id}_3$, $\phi_{\alpha,\beta}(1) = (123)$, $\phi_{\alpha,\beta}(2) = (132)$.

Then, $S = B_\alpha \,\dot\cup\, B_\beta$ endowed with the operation given by

$$\forall a \in B_\alpha, b \in B_\beta \quad \begin{aligned} a + b &:= \phi_{\alpha,\beta}(a) + \phi_{\beta,\beta}(b) \\ a \circ b &:= \phi_{\alpha,\beta}(a) \circ \phi_{\beta,\beta}(b) \end{aligned}$$

is a (not trivial) dual weak brace.

Homomorphisms between skew braces

Problem

Finding homomorphism between skew braces for constructing dual weak braces.

This problem already emerged in literature, such as:

- **[Cedó, 2018], [Vendramin, 2019]** pose the problem of computing the automorphism groups of skew braces of size p^n
- **[Zenouz, 2019]** determines the automorphism group of skew braces of order $p > 3$
- **[Puljić, Smoktunowicz, Zenouz, 2022]** describe $\mathbb{F}_p$-braces of cardinality p^4 which are not right nilpotent
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Strong semilattice of solutions

Theorem (Catino, Colazzo, Stefanelli, 2021)

Let Y be a (lower) semilattice, $\{r_\alpha \mid \alpha \in Y\}$ a family of disjoint solutions on X_α indexed by Y such that for each $\alpha \geq \beta$ there is a map $\phi_{\alpha,\beta} : X_\alpha \to X_\beta$. Let $X = \bigcup \{X_\alpha \mid \alpha \in Y\}$ and $r : X \times X \longrightarrow X \times X$ be the map defined as

$$r(x, y) := r_{\alpha\beta}\left(\phi_{\alpha,\alpha\beta}(x), \phi_{\beta,\alpha\beta}(y)\right),$$

for all $x \in X_\alpha$ and $y \in X_\beta$. If the following conditions are satisfied:

1. $\phi_{\alpha,\alpha}$ is the identity map of X_α, for every $\alpha \in Y$,
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3. $(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta})\, r_\alpha = r_\beta\, (\phi_{\alpha,\beta} \times \phi_{\alpha,\beta})$, for all $\alpha, \beta \in Y$ such that $\alpha \geq \beta$,

then r is a solution on X, called strong semilattice of the solutions r_α.

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The solutions coming from dual weak braces

Theorem

Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace. Then, the solution r associated to S is the strong semilattice of the bijective non-degenerate solutions r_α associated to each skew brace B_α.

Corollary

Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a *finite* dual weak brace and r the solution associated to S. Then, $r^{2k+1} = r$ with $2k = \operatorname{lcm}\{p(r_\alpha) \mid \alpha \in Y\}$.

Consequently, the solution r associated to a finite dual weak brace $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ is cubic, i.e., $r^3 = r$, if and only if r_α is involutive, i.e., $r_\alpha^2 = \mathrm{id}_{B_\alpha \times B_\alpha}$, that is each B_α is a brace.

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Ideals of dual weak braces

Definition

A subset I of a dual weak brace S is an *ideal* of $(S, +, \circ)$ if

1. I is a normal subsemigroup of $(S, +)$,
2. $\lambda_a(I) \subseteq I$, for every $a \in S$,
3. I is a normal subsemigroup of $(S, \circ)$.

If I is an ideal, the relation $\sim_I$ on S given by

$$\forall\ a, b \in S \quad a \sim_I b \iff a - a = b - b \text{ and } -a + b \in I,$$

is a congruence of $(S, +, \circ)$.

Example

The set $\mathrm{Soc}(S) = \{a \mid a \in S,\ \forall\ b \in S \quad a + b = a \circ b \quad \text{and} \quad a + b = b + a\}$ is an ideal of any dual weak brace $(S, +, \circ)$ called *socle* of S.

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Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace, I_α an ideal of each skew brace B_α, and set $\psi_{\alpha,\beta} := \phi_{\alpha,\beta}|_{I_\alpha}$, for all $\alpha \geq \beta$.
If $\phi_{\alpha,\beta}(I_\alpha) \subseteq I_\beta$, for any $\alpha > \beta$, then $I = [Y; I_\alpha; \psi_{\alpha,\beta}]$ is an ideal of S and, conversely, every ideal of S is of this form.

Question

Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace.

$$\text{Is} \quad \mathrm{Soc}(S) = [Y; \mathrm{Soc}(B_\alpha); \psi_{\alpha,\beta}]?$$

In general, the answer is negative. For instance, in the example with $Y = \{\alpha, \beta\}$, with $\alpha > \beta$, $B_\alpha = C_3$, and $B_\beta = \mathrm{Sym}_3$, we have that $\mathrm{Soc}(B_\alpha) \cup \mathrm{Soc}(B_\beta)$ is not an ideal of S.

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Thank you!