Gauge fixing in 3d gravity and classical dynamical r-matrices

Juan Carlos Morales Parra
Joint with Prof. Bernd Schroers

Heriot-Watt University
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Outline

1. Chern-Simons formulation of 3d gravity
2. Classical r-matrices: Phase space of 3d (CS) gravity
3. Classical dynamical r-matrices: Gauge fixing in 3d (CS) gravity
4. Further results

Pure gravity in 3 dimensions

2 key observations about 3d GR

1. Geometry: The Weyl tensor is identically zero.
2. Einstein's equation: The equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$ reduces to $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$.

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Local model spacetimes and isometry groups ($G_{\Lambda,c}$)

Λ	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
0	$\mathbf{E}^3 = \mathrm{ISO}(3)/\mathrm{SO}(3)$	$\mathbf{M}^{2+1} = \mathrm{ISO}(2,1)/\mathrm{SO}(2,1)$
> 0	$\mathbf{S}^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$	$\mathbf{dS}^{2+1} = \mathrm{SO}(3,1)/\mathrm{SO}(2,1)$
< 0	$\mathbf{H}^3 = \mathrm{SO}(3,1)/\mathrm{SO}(3)$	$\mathbf{AdS}^{2+1} = \mathrm{SO}(2,2)/\mathrm{SO}(2,1)$

Chern-Simons formulation of pure 3d gravity

Chern-Simons formulation

With the combined connection $A = e^a P_a + \omega^a J_a$, the Chern-Simons action

$$S_{CS}(A) = \frac{k}{4\pi} \int_{\mathcal{M}^3} \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle \quad with \quad k = \frac{1}{4\pi G_{Newton}}$$

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Lie algebra $\mathfrak{g}_{\Lambda,c}$

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad \text{and} \quad [P_a, P_b] = (\underbrace{-c^2\Lambda}_{\lambda})\epsilon_{abc} J^c$$

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Ad-invariant (standard) symmetric bilinear form

$$\langle J_a, J_b \rangle = 0, \quad \langle J_a, P_b \rangle = c^2 \eta_{ab} \quad \text{and} \quad \langle P_a, P_b \rangle = 0$$

Phase space for 3d (CS) gravity?

Phase space $\mathcal{P}$ of CS formulation of 3d gravity

For $\mathcal{M}_3 \cong \mathbb{R} \times S_{g,n}$, the moduli space is parametrized by the set of holonomies along the generators of $\pi_1(S_{g,n})$ such that $[A_g, B_g^{-1}] \cdots [A_1, B_1^{-1}] \cdot M_n \cdots M_1 = 1$, modulo global conjugation.

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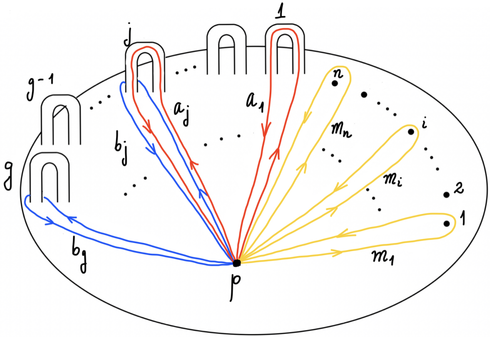
Graphically



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- $F(M_i, A_j, B_j) = \overleftarrow{\prod}_{j=1}^{g}[B_j, A_j^{-1}]\overleftarrow{\prod}_{i=1}^{n} M_i$.

Poisson structure over $\mathcal{P}$

Fock-Rosly (Alekseev) approach to Atiyah-Bott [92]

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$$\Pi^r_{FR} = \sum_{i=1}^{n} r^{\alpha\beta} \left(\frac{1}{2} R^{M_i}_{\alpha} \wedge R^{M_i}_{\beta} + \frac{1}{2} L^{M_i}_{\alpha} \wedge L^{M_i}_{\beta} + R^{M_i}_{\alpha} \wedge L^{M_i}_{\beta} \right) + \cdots$$

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i.e. $\{F, G\} = \Pi^r_{FR}(dF, dG)$ with $F, G \in \mathcal{C}^\infty(G^{n+2g}_{\Lambda,c})$.

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Key ingredient

$r = r^{ab} T_a \otimes T_b \in \mathfrak{g}^2_{\Lambda,c}$ is a **classical** r-matrix.

First constraining $\rightarrow$ Second Quantization

Gauge fixing the F=1 constraint

Using 6 auxiliary gauge constraints over $\mathcal{P}_{\text{ext}}$ (e.g. just involving M_1 and M_2), the Dirac bracket defines a Poisson space $(\mathcal{P}_{\text{ext}}^{GF}, \Pi_{FR}^{r_d})$ with

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and Poisson structure given by $F, G \in \mathcal{C}^{\infty}(G^{n+2g-2}_{\Lambda,c})$

$$\{\alpha,\psi\}_D = 0, \qquad \{F,G\}_D = \Pi^{r_d}_{FR}(dF,dG),$$
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$r_d = r^{\alpha,\beta}(\alpha, \beta) T_\alpha \otimes T_\beta \in \text{Mer}(\mathbb{R}^2) \otimes (\mathfrak{g}_{\Lambda,c}^2)$ is a **dynamical classical** r-matrix

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Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra and $K \in (S^2\mathfrak{g})^{\mathfrak{g}}$. A *classical dynamical* $(\mathfrak{g}, \mathfrak{h}, K)$*-r-matrix* is an $\mathfrak{h}$-equivariant meromorphic function

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$$\sum_i \left(x_i^{(1)} \frac{\partial r_d^{23}}{\partial x_i} - x_i^{(2)} \frac{\partial r_d^{13}}{\partial x_i} + x_i^{(3)} \frac{\partial r_d^{12}}{\partial x_i} \right) + [r_d^{12}, r_d^{13}] + [r_d^{12}, r_d^{23}] + [r_d^{13}, r_d^{23}] = 0 \tag{2}$$

where x_i is a basis of $\mathfrak{h}$ (i.e. a complete set of coordinate functions for $\mathfrak{h}^*$).

Classical dynamical r-matrices for $\mathfrak{g}_{\Lambda,c}$

Theorem. Let $\mathfrak{h}_{\Lambda,c}$ be a Lie subalgebra of $\mathfrak{g}_{\Lambda,c}$, with basis $\{\boldsymbol{\alpha}, \boldsymbol{\psi}\}$. A function $r_d \in \mathrm{Mer}(\mathfrak{h}^*_{\Lambda,c}, \mathfrak{g}_{\Lambda,c} \otimes \mathfrak{g}_{\Lambda,c})$ given by

$$
\begin{aligned}
r_d(\boldsymbol{\alpha}, \boldsymbol{\psi}) = \frac{1}{2}(J_a \otimes P^a + P_a \otimes J^a) + J^a \otimes A(\boldsymbol{\alpha}, \boldsymbol{\psi}) J_a \\
+ P^a \otimes B(\boldsymbol{\alpha}, \boldsymbol{\psi}) J_a - B(\boldsymbol{\alpha}, \boldsymbol{\psi}) J_a \otimes P^a + P^a \otimes C(\boldsymbol{\alpha}, \boldsymbol{\psi}) P_a
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is a classical dynamical r-matrix if and only if

$$\frac{1}{2}\mathrm{tr}(A^2) - \frac{\lambda}{2}[\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)] + \mathrm{div}_{\boldsymbol{\alpha}}(v^A) = \mu\lambda$$
$$\mathrm{tr}(CB) + \mathrm{div}_{\boldsymbol{\psi}}(v^C) = 0$$
$$A(B + B^t) - (B^t - \mathrm{tr}(B)\mathrm{id})(\lambda C - A) + \mathrm{tr}(AB)\mathrm{id}$$
$$- \mathrm{curl}_{\boldsymbol{\alpha}}(B^t) + \mathrm{grad}_{\boldsymbol{\psi}}(v^A) = 0$$

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- Dynamical generalizations of Fock-Rosly spaces.

Thank You!

Any Questions? Please ask!