Fundamental Superalgebras in PI Theory

(joint work with A. Giambruno and E. Spinelli)

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Polinomial identities

Fix a field F. Some notations:

- A is an associative algebra,
- $X := \{x_1, x_2, \dots\}$ is a countable set,
- $F\langle X\rangle$ is the free associative algebra over F generated by X.

Definition

An element $f(x_1, \dots, x_n)$ of $F\langle X\rangle$ is a **polynomial identity** (or a *PI*) for A if $f(a_1, \dots, a_n) = 0_A$ for every $a_1, \dots, a_n \in A$.
A is a **PI-algebra** if A satisfies a non-trivial polynomial identity $f \neq 0_{F\langle X\rangle}$.
Let $Id(A) := \{f \mid f \in F\langle X\rangle,\ f\ PI \text{ for } A\}$.

Some notations:

- A is an associative superalgebra (i.e. $A = A^{(0)} \oplus A^{(1)}$ s.t. $A^{(i)}A^{(j)} \subseteq A^{(i+j)}$ for every $i, j \in \mathbb{Z}_2$),
- $Y := \{y_1, y_2, \ldots\}$ and $Z := \{z_1, z_2, \ldots\}$ are (disjoint) countable sets,
- $F\langle Y \cup Z\rangle$ is the free associative algebra over F generated by $Y \cup Z$ ($\rightsquigarrow$ structure of superalgebra: $\deg y_i = 0$, $\deg z_i = 1$ for every $i \geq 1$).

Definition

An element $f(y_1, \ldots, y_m, z_1, \ldots, z_n)$ of $F\langle Y \cup Z\rangle$ is a $\mathbb{Z}_2$**-graded polynomial identity** or a **superidentity** ($f \equiv 0$) for a superalgebra $A = A^{(0)} \oplus A^{(1)}$ if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0_A$ for every $a_1, \ldots, a_m \in A^{(0)}$ and $b_1, \ldots, b_n \in A^{(1)}$.

Graded polynomial identities

1 Introduction

In what follows, F will be an algebraically closed field of characteristic zero.
A central object in the theory is:

$$\mathbf{Id}_2(A)$$

the set of all $\mathbb{Z}_2$-graded polynomial identities satisfied by A

Fact

$Id_2(A)$ is a T_2**-ideal**, i.e. a two-sided ideal of the free superalgebra stable under every graded endomorphism of the free superalgebra, completely determined by the *multilinear polynomials* it contains.

Specht Problem

One of the main questions in PI theory is

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A positive solutions was given by Kemer in 1987.

One of the main steps of the proof is the following

Kemer's Representability Theorem

Let *A* be a *PI*-finitely generated (super)algebra over a field *F* of characteristic zero. Then there exists a finite-dimensional (super)algebra over a field extension of *F* which has the same (graded) polynomial identities of *A*.

One of the tools to prove Kemer's Representability Theorem was the introduction of the so called **fundamental (super)algebras**. The main reason lies on one of their properties.

Any finite-dimensional (super)algebra has the same (graded) identities as a finite direct sum of fundamental (super)algebras.

Fix a finite-dimensional algebra A.

Wedderburn Malcev Decomposition

$$A = A_{ss} + J(A),$$

where $J(A)$ is the **Jacobson radical** (which is a nilpotent ideal of nilpotency index $s_A + 1$) and where A_{ss} is a **maximal semisimple subalgebra** of A. A_{ss} can be written as a direct sum of simple algebras: $A_{ss} = A_1 \oplus \ldots \oplus A_n$.

Algebra index

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Definition

The $t\text{-}s(A) := (dim_F(A_{ss}), s_A)$ is said to be the **algebra index** of A.

Understanding Fundamental Algebras

Remark: we can always restrict our attention on evaluations on a given fixed basis of our algebra.

Facts

If f is alternating in $d > dim_F(A)$ variables, then $f \in Id(A)$.

Now assume that you have a polynomial f alternating in an arbitrary number of sets each with $d \leq dim_F(A_{ss})$ variables. It might be that there exists a non-zero evaluation in A_{ss}.
Now if you allow some of these sets (say s) to be of cardinality $dim_F(A_{ss}) + 1$, you *might* find a non-zero evaluation on A only if you require that $s \leq s_A$, where $s_A + 1$ is the nilpotency index of $J(A)$.
So in some sense a *fundamental algebra* is an "extreme" algebra, in the sense that it realizes the maximal possible number of alternations.

(almost a) Definition

A is **fundamental** if there exists a polynomial $f \notin Id(A)$ which is alternating in an arbitrary number of sets, each of cardinality $dim_F(A_{ss})$, and which is also alternating in s_A sets of cardinality $dim_F(A_{ss}) + 1$.

Fix a finite-dimensional superalgebra A.

Wedderburn Malcev Decomposition (graded version)

$$A = A_{ss} + J(A),$$

where $J(A)$ is the **Jacobson radical** (which is a homogeneous nilpotent ideal) and where A_{ss} is a **maximal semisimple subalgebra** of A having an induced $\mathbb{Z}_2$-grading. A_{ss} can be written as a direct sum of graded simple algebras: $A_{ss} = A_1 \oplus \ldots \oplus A_n$.

Superindices

4 Fundamental Superalgebras

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Definition

The **first algebra superindex** of A is the pair $(t_{A,0}, t_{A,1}) := (\dim_F A_{ss}^{(0)}, \dim_F A_{ss}^{(1)})$, whereas its **second algebra superindex** is s_A, where $s_A + 1$ is the nilpotency index of $J(A)$.
The triple $t\text{-}s_2(A) := (t_{A,0}, t_{A,1}; s_A)$ is said to be the **algebra superindex** of A.

Superindices

If t_0, t_1, ν are integers, an element $f \in F\langle Y \cup Z\rangle$ is said to be **ν-fold (t_0, t_1)-alternating** if, for all $i \in [1, \nu] := \{1, 2, \ldots, \nu\}$, there exist sets of variables $Y_i \subseteq Y$ and $Z_i \subseteq Z$ with $|Y_i| = t_0$ and $|Z_i| = t_1$ such that f is alternating in each Y_i and in each Z_i.

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Definition

The **first Kemer superindex** of $I := Id_2(A)$ is the maximum (if exists), in the lexicographic order, of the pairs of numbers (t_0, t_1) such that, for all $\nu \in \mathbb{N}$, there is an element $f \in F\langle Y \cup Z\rangle \setminus I$ which is ν-fold (t_0, t_1)-alternating.
If (t_0, t_1) is the first Kemer superindex of I, its **second Kemer superindex** is the maximum integer $s = a + b$ for which, for every integer ν, there exists a ν-fold (t_0, t_1)-alternating polynomial $f \in F\langle Y \cup Z\rangle \setminus I$ which is a-fold alternating in layers of degree 0 variables with $t_0 + 1$ elements and b-fold alternating in layers of degree 1 variables with $t_1 + 1$ elements.
The **Kemer superindex** of a superalgebra A is the triple $\mathrm{Ind}_2(A) := (t_0, t_1; s)$.

Main Definition

Fact

$$Ind_2(A) \leq t\text{-}s_2(A)$$

in the lexicographic order.

Definition

A is **fundamental** if and only if $Ind_2(A) = t\text{-}s_2(A)$.

Recap on Simple Superalgebras

A $\mathbb{Z}_2$-grading on M_m is called **elementary** if there exists an m-tuple $(g_1, \ldots, g_m) \in \mathbb{Z}_2^m$ such that $e_{ij} \in M_m^{(g)}$ if, and only if, $g = g_i - g_j$. Equivalently, we can define a map $\alpha : [1, m] \longrightarrow \mathbb{Z}_2$ inducing a grading on M_m setting the degree of e_{ij} equal to $\alpha(i) - \alpha(j)$.

Let A be a simple superalgebra. A is (isomorphic to) a superalgebra of the following type

(a) $M_{k,l} := M_{k+l}$ with $k \geq l \geq 0$, $k \neq 0$, endowed with the grading induced by the $(k + l)$-tuple $(\underbrace{0, \ldots, 0}_{k \text{ times}}, \underbrace{1, \ldots, 1}_{l \text{ times}})$;

(b) $M_m + tM_m$, where $t^2 = 1_F$, with grading (M_m, tM_m). It can be realized as the homogeneous subalgebra $\left\{ \begin{pmatrix} C & D \\ D & C \end{pmatrix} \right\}$ of the full matrix algebra M_{2m} with the grading induced by the $(2m)$-tuple $(\underbrace{0, \ldots, 0}_{m \text{ times}}, \underbrace{1, \ldots, 1}_{m \text{ times}})$.

Examples of Fundamental Superalgebras

Proposition

Every finite-dimensional simple superalgebra is fundamental.

Why? Just to have an idea, let us look an example.
Consider $M_2 + tM_2$. It is clear that t-$s_2(M_2 + tM_2) = (4, 4; 0)$.

$$e_{1,1}\tilde{e}_{1,1}e_{1,1}\tilde{e}_{1,2}e_{2,2}\tilde{e}_{2,2}e_{2,2}\tilde{e}_{2,1}e_{1,1} \dots e_{1,1}\overline{te_{1,1}}e_{1,1}\overline{te_{1,2}}e_{2,2}\overline{te_{2,2}}e_{2,2}\overline{te_{2,1}}e_{1,1} \dots,$$

where we have used this notation: $\bar{x}_1 \dots \bar{x}_n := \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \dots x_{\sigma(n)}$ (or with other fancy symbols).
From this product,which is obvious non-zero (equal to $e_{1,1}$), it can be built a graded polynomial, ν-fold $(4, 4)$-alternating, outside $Id_2(M_2 + tM_2)$ with this non-zero evaluation.

Let $((A_1, \alpha_1) \dots, (A_n, \alpha_n))$ be a sequence of simple superalgebras.

$$s_j := \begin{cases} k_j + l_j & \text{if } A_j \cong M_{k_j, l_j}, \\ 2n_j & \text{if } A_j \cong M_{n_j} + tM_{n_j} \end{cases}, \quad s'_j := \begin{cases} k_j & \text{if } A_j \cong M_{k_j, l_j}, \\ n_j & \text{if } A_j \cong M_{n_j} + tM_{n_j} \end{cases},$$

and set $\eta_0 := 0$, $\eta_j := \sum_{i=1}^{j} s_i$ and $Bl_j := [\eta_{j-1} + 1, \eta_j]$. In particular $\eta_n = \sum_{i=1}^{n} s_i$.
Let $\mathbf{U} := \{(a_{ij})_{i,j \in [1,n]} \mid a_{ij} \in M_{s_i \times s_j} \text{ if } 1 \leq i \leq j \leq n \text{ and } a_{ij} = 0_{M_{s_i \times s_j}} \text{ otherwise}\} \subseteq M_{\eta_n}$
(the upper block triangular matrix algebra of size $s_1, \dots, s_n$).

Upper block triangular matrix algebras

Finally, let us define

$UT(A_1, \dots, A_n) := \left\{ (a_{ij}) \in \mathbf{U} \mid a_{kk} = \begin{pmatrix} C & D \\ D & C \end{pmatrix}, \ C, D \in M_{s'_k} \ \forall k \in \Gamma_1 \right\}$. Let

$$\alpha : [1, \eta_n] \longrightarrow \mathbb{Z}_2, \qquad i \longmapsto \alpha_k(i - \eta_{k-1})$$

and, for any n-tuple $\tilde{g} := (g_1, \dots, g_n) \in \mathbb{Z}_2^n$,

$$\alpha_{\tilde{g}} : [1, \eta_n] \longrightarrow \mathbb{Z}_2, \qquad i \longmapsto g_k + \alpha(i),$$

where $k \in [1, n]$ is the (unique) integer such that $i \in Bl_k$.
Denote any such a $\mathbb{Z}_2$-graded algebra (regardless of $\tilde{g}$) by $UT_{\mathbb{Z}_2}(A_1, \dots, A_n)$.

Why are we studying these algebras?

Theorem [O.M. Di Vincenzo, V.R.T. da Silva, E. Spinelli]

A variety of $\mathbb{Z}_2$-graded PI-algebras of finite basic rank is minimal of superexponent $d \geq 2$ if, and only if, it is generated by a $\mathbb{Z}_2$-graded algebra $UT_{\mathbb{Z}_2}(A_1, \ldots, A_n)$ satisfying $\dim_F(A_1 \oplus \ldots \oplus A_n) = d$.

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We have proved:

Theorem [A. Giambruno, P, E. Spinelli]

The superalgebras $UT_{\mathbb{Z}_2}(A_1, \ldots, A_n)$ are fundamental.

Upper block triangular matrix algebras with identification

Let $(A_1, \ldots, A_n)$ be a sequence of simple superalgebras for which there exist $1 \leq i < j \leq n$ such that $A_i \cong A_j$. Consider the superalgebra $B := UT_{\mathbb{Z}_2}(A_1, \ldots, A_n)$ with grading defined by the map $\alpha_{\tilde{g}}$, and its subalgebra, $A := UT^{(i,j)}_{\mathbb{Z}_2}(A_1, \ldots, A_n)$, obtained from B by *"gluing"* A_i and A_j. We notice that $J(A) = J(B)$, whereas $\dim_F A_{ss} = \dim_F B_{ss} - \dim_F A_i$.

Theorem [A. Giambruno, P, E. Spinelli]

The superalgebra A is fundamental if, and only if,

1. either $j \neq i + 1$;
2. or $j = i + 1$ and $A_i \cong M_{n_i} + tM_{n_i}$;
3. or $j = i + 1$, $A_i \cong M_{k_i, l_i}$ and $g_{i+1} = g_i + 1$.

Let $P_n^{\mathbb{Z}_2}$ is the space of multilinear polynomials of degree n of $F\langle Y \cup Z\rangle$ in the variables $y_1, \dots, y_n, z_1, \dots, z_n$.

Facts

- the *hyperoctahedral group* $H_n = \mathbb{Z}_2 \wr S_n$ (namely the wreath product of $\mathbb{Z}_2$ and the symmetric group S_n) acts on $P_n^{\mathbb{Z}_2}$,
- $P_n^{\mathbb{Z}_2} \cap \mathrm{Id}_2(A)$ is invariant under this action,
- the space $P_n^{\mathbb{Z}_2}(A) := \frac{P_n^{\mathbb{Z}_2}}{P_n^{\mathbb{Z}_2} \cap \mathrm{Id}_2(A)}$ has a structure of left H_n-module, whose character, $\chi_n^{\mathbb{Z}_2}(A)$, is called the **n-th $\mathbb{Z}_2$-graded cocharacter** of A,
- $\{\text{irreducible } H_n\text{-representations}\}/\sim \ \xleftrightarrow{1:1} \{(\lambda, \mu),\ \lambda \vdash r, \mu \vdash n-r, r \in [0, n]\}$.

A characterization through cocharacters

So, by the previous facts, it makes sense to write:

$$\chi_n^{\mathbb{Z}_2}(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu}.$$

Theorem [A. Giambruno, P, E. Spinelli]

Let A be a finite-dimensional superalgebra with algebra superindex $(d_0, d_1; s)$. Then A is fundamental if, and only if, for any n large enough, there exist $r \in [0, n]$, $\lambda := (\lambda_1, \ldots, \lambda_t) \vdash r$ and $\mu := (\mu_1, \ldots, \mu_u) \vdash n - r$ with $\lambda_{d_0+1} + \cdots + \lambda_t = a$ and $\mu_{d_1+1} \cdots + \mu_u = b$ such that $a + b = s$ and $m_{\lambda,\mu} \neq 0$ in $\chi_n^{\mathbb{Z}_2}(A)$.

Thank you for listening!
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