On central nilpotency and solubity of skew left braces and solutions of the Yang-Baxter equation

Vicent Pérez Calabuig*
Universitat de València

*Joint work with Adolfo Ballester Bolinches and Ramon Esteban Romero

Groups, rings and the Yang-Baxter equation
Blankenberge, 2023

- A *skew left brace* $(B, +, \cdot)$ is a set endowed with two group structures, $(B, +)$ and $(B, \cdot)$, such that

$$a(b + c) = ab - a + ac \quad \forall\, a, b, c \in B.$$

- Two of the most studied properties of non-degenerate solutions: decomposability and multipermutability.

  - P. Etingof, T. Schedler and A. Soloviev (1999), *Set theoretical solutions to the Yang-Baxter Equation*, Duke Math. J, 100(2), 169–209.

- Algebraic framework for these properties: solubility and central nilpotency of skew left braces.

Preliminaries

- Solutions: non-degenerate set-theoretic solutions of the YBE.
- Braces: skew left braces.
- Let $\mathfrak{X}$ be a class of groups: B is a *brace of $\mathfrak{X}$-type* if $(B,+) \in \mathfrak{X}$. For example, Rump's left braces are braces of abelian type.
- $\lambda\colon (B,\cdot) \to \mathrm{Aut}(B,+)$, $\lambda_a(b) = -a + ab$.
- $*$-product: $a * b = \lambda_a(b) - b = -a + ab - b$.
- A *trivial brace* B is a brace such that $a + b = ab$ for every $a, b \in B$; equivalently, $a * b = 0$ for every $a, b \in B$.
- Given two subsets $X, Y \subseteq B$, $X * Y = \langle x * y : x \in X, y \in Y \rangle_+$.

- Algebraic substructures of a brace:
  - A *subbrace* S of B is a subgroup of the additive group which is also a subgroup of the multiplicative group.
  - A *left ideal* L of B is a λ-invariant subbrace. If $(L,+) \trianglelefteq (B,+)$ then, L is a *strong left ideal*.
  - An *ideal* I of B is a strong left ideal, such that $(I,\cdot) \trianglelefteq (B,\cdot)$.
- $\mathrm{Fix}(B) := \{a \in B : \lambda_b(a) = a, \ \forall b \in B\}$ is a left ideal.
- $\mathrm{Soc}(B) := \{a \in B : ab = a + b = b + a, \ \forall b \in B\} = \mathrm{Ker}\,\lambda \cap \mathrm{Z}(B,+)$ is an ideal.

- Algebraic substructures of a brace:
  - A *subbrace* S of B is a subgroup of the additive group which is also a subgroup of the multiplicative group.
  - A *left ideal* L of B is a λ-invariant subbrace. If $(L,+) \trianglelefteq (B,+)$ then, L is a *strong left ideal*.
  - An *ideal* I of B is a strong left ideal, such that $(I,\cdot) \trianglelefteq (B,\cdot)$.

- $\mathrm{Fix}(B) := \{a \in B : \lambda_b(a) = a, \ \forall b \in B\}$ is a left ideal.

- $\mathrm{Soc}(B) := \{a \in B : ab = a + b = b + a, \ \forall b \in B\} = \mathrm{Ker}\,\lambda \cap \mathrm{Z}(B,+)$ is an ideal.

- Given I, J ideals of a brace B with $J \subseteq I$, we say that I/J is a *chief factor* if I/J is a minimal ideal of B/J.
- An ideal series of a brace B is a sequence of ideals of B:

$$I_0 = 0 \subseteq I_1 \subseteq \ldots \subseteq I_n = B$$

- A *chief series* is an ideal series such that I_k/I_{k-1} is a chief factor, for every $1 \leq k \leq n$.

Soluble braces and decomposable solutions

- A solution (X, r) is said to be *decomposable* if there exists a non-trivial partition $X = X_1 \cup X_2$ such that $r(X_i \times X_j) = X_j \times X_i$ for every $i, j \in \{1, 2\}$.

Proposition

Let B be a brace and (B, r_B) its associated solution. Assume that B has an strong left ideal I. Then, (B, r_B) is decomposable as $B = I \cup (B \setminus I)$.

The richer is the ideal structure of a brace the more decomposable is its associated solution.

Soluble braces and decomposable solutions

- A solution (X, r) is said to be *decomposable* if there exists a non-trivial partition $X = X_1 \cup X_2$ such that $r(X_i \times X_j) = X_j \times X_i$ for every $i, j \in \{1, 2\}$.

Proposition

Let B be a brace and (B, r_B) its associated solution. Assume that B has an strong left ideal I. Then, (B, r_B) is decomposable as $B = I \cup (B \setminus I)$.

The richer is the ideal structure of a brace the more decomposable is its associated solution.

Soluble braces and decomposable solutions

- A solution (X, r) is said to be *decomposable* if there exists a non-trivial partition $X = X_1 \cup X_2$ such that $r(X_i \times X_j) = X_j \times X_i$ for every $i, j \in \{1, 2\}$.

Proposition

Let B be a brace and (B, r_B) its associated solution. Assume that B has an strong left ideal I. Then, (B, r_B) is decomposable as $B = I \cup (B \setminus I)$.

The richer is the ideal structure of a brace the more decomposable is its associated solution.

Soluble braces

Let B be a brace.

- A previous definition: consider the series of iterated star products

$$B_1 = B \supseteq B_2 = B_1 * B_1 \supseteq \ldots \supseteq B_n = B_{n-1} * B_{n-1} \supseteq \ldots$$

  Then, B is soluble if there exists $n_0 \in \mathbb{N}$ such that $B_{n_0} = 0$.
- Every trivial brace is soluble $\Rightarrow$ every group as a trivial brace is soluble.
- It is convenient to have a useful definition of commutator of ideals
  - D. Bourn, A. Facchini and M. Pompili (2022), *Aspects of the category SKB of skew braces*, Comm. Algebra, 51(5), 2129–2143.

Soluble braces

Let B be a brace.

- A previous definition: consider the series of iterated star products

$$B_1 = B \supseteq B_2 = B_1 * B_1 \supseteq \ldots \supseteq B_n = B_{n-1} * B_{n-1} \supseteq \ldots$$

  Then, B is soluble if there exists $n_0 \in \mathbb{N}$ such that $B_{n_0} = 0$.

- Every trivial brace is soluble $\Rightarrow$ every group as a trivial brace is soluble.
- It is convenient to have a useful definition of commutator of ideals
  - D. Bourn, A. Facchini and M. Pompili (2022), *Aspects of the category SKB of skew braces*, Comm. Algebra, 51(5), 2129–2143.

Soluble braces

Let B be a brace.

- A previous definition: consider the series of iterated star products

  $$B_1 = B \supseteq B_2 = B_1 * B_1 \supseteq \ldots \supseteq B_n = B_{n-1} * B_{n-1} \supseteq \ldots$$

  Then, B is soluble if there exists $n_0 \in \mathbb{N}$ such that $B_{n_0} = 0$.
- Every trivial brace is soluble $\Rightarrow$ every group as a trivial brace is soluble.
- It is convenient to have a useful definition of commutator of ideals
  - D. Bourn, A. Facchini and M. Pompili (2022), *Aspects of the category SKB of skew braces*, Comm. Algebra, 51(5), 2129–2143.

Definition

Let B be a brace and let I, J be ideals of B. The *commutator* $[I, J]$ *in* B is defined as

$$[I, J]_B := \langle [I, J]_+, [I, J]_\cdot, \{ij - (i + j) : i \in I,\ j \in J\} \rangle$$

Soluble braces

Let B be a brace. We define:

- $\partial(B) = [B, B]_B$, the *commutator ideal or derived ideal* of B.
- B is said to be abelian if $\partial(B) = 0$; equivalently, B is a trivial brace with abelian group structure.
- A derived series of B:

$$\begin{aligned} B = \partial_0(B) \supseteq \partial_1(B) = \partial(B) \supseteq \dots \\ \supseteq \partial_n(B) = [\partial_{n-1}(B), \partial_{n-1}(B)]_B \supseteq \dots \end{aligned}$$

- B is said to be *soluble* if there exists n_0 such that $\partial_{n_0}(B) = 0$.

Definition

Let B be a brace and let I, J be ideals of B. The *commutator* $[I, J]$ *in* B is defined as

$$[I, J]_B := \langle [I, J]_+, [I, J]_\cdot, \{ij - (i + j) : i \in I,\ j \in J\} \rangle$$

Soluble braces

Let B be a brace. We define:

- $\partial(B) = [B, B]_B$, the *commutator ideal or derived ideal* of B.
- B is said to be abelian if $\partial(B) = 0$; equivalently, B is a trivial brace with abelian group structure.
- A derived series of B:

$$B = \partial_0(B) \supseteq \partial_1(B) = \partial(B) \supseteq \dots$$
$$\supseteq \partial_n(B) = [\partial_{n-1}(B), \partial_{n-1}(B)]_B \supseteq \dots$$

- B is said to be *soluble* if there exists n_0 such that $\partial_{n_0}(B) = 0$.

Proposition

A brace B is soluble if, and only if, it has an abelian series, that is a decreasing sequence of ideals

$$I_0 = B \supseteq I_1 \supseteq \ldots \supseteq I_n = 0$$

such that I_k/I_{k+1} is abelian for every $0 \leq k \leq n-1$. In such a case, $\partial_k(B) \subseteq I_k$, for every $0 \leq k \leq n$.
Therefore, the derived length of a soluble brace B is the smallest length of an abelian series of B.

Theorem

Let B be a soluble brace with a chief series. Then, each chief factor is abelian and it is either complemented or included in the intersection of all maximal subbraces.

Moreover, if B is finite, then

- *Each chief factor is as a trivial brace isomorphic to an elementary abelian p-group for some prime p*
- *Every maximal subbrace has a prime power index as a subgroup of both $(B, +)$ and $(B, \cdot)$.*

Multidecomposable solutions

Definition

Let (X, r) be a solution and let $\mathcal{P} = \{X_i \subseteq X : i \in I\}$ be a (uniform) partition of X. We say that (X, r) is *(uniformly) $\mathcal{P}$-decomposable* if $r(X_i \times X_j) = X_j \times X_i$ for every $i, j \in I$.

Definition

Let (X, r) be a solution and let $X_n \subseteq X_{n-1} \subseteq \ldots X_1 \subseteq X_0 = X$ be a finite descending sequence of subsets of X with $|X_n| = 1$. Assume that for every $0 \leq i \leq n-1$, there exists a (uniform) partition $\mathcal{P}_i$ of X_i such that $X_{i+1} \in \mathcal{P}_i$ and $(X_i, r|_{X_i \times X_i})$ is (uniformly) $\mathcal{P}_i$-decomposable. Then, we say that (X, r) is *(uniformly) multidecomposable* of level n.

Multidecomposable solutions

Definition

Let (X, r) be a solution and let $\mathcal{P} = \{X_i \subseteq X : i \in I\}$ be a (uniform) partition of X. We say that (X, r) is *(uniformly) $\mathcal{P}$-decomposable* if $r(X_i \times X_j) = X_j \times X_i$ for every $i, j \in I$.

Definition

Let (X, r) be a solution and let $X_n \subseteq X_{n-1} \subseteq \ldots X_1 \subseteq X_0 = X$ be a finite descending sequence of subsets of X with $|X_n| = 1$. Assume that for every $0 \leq i \leq n-1$, there exists a (uniform) partition $\mathcal{P}_i$ of X_i such that $X_{i+1} \in \mathcal{P}_i$ and $(X_i, r|_{X_i \times X_i})$ is (uniformly) $\mathcal{P}_i$-decomposable. Then, we say that (X, r) is *(uniformly) multidecomposable* of level n.

Theorem

Let B be a soluble brace. Then its associated solution (B, r_B) is uniformly multidecomposable.

Theorem

Let (X, r) be a solution such that $G(X, r)$ is a soluble brace. Assume that $G(X, r)$ has an abelian descending series

$$G(X, r) = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n = 0$$

such that $X \cap I_{n-1}$ is not empty. Then (X, r) is multidecomposable.

Central nilpotency

$$\text{Central nilpotency} \to \text{Strong nilpotency} \to \begin{cases} \text{Right nilpotency} \\ \\ \text{Left nilpotency} \end{cases}$$

Centrally nilpotent braces

Let B be a brace:

- $\zeta(B) = \mathrm{Soc}(B) \cap \mathrm{Fix}(B) =$

$$= \{a \in B : b + a = a + b = ab = ba, \ \forall b \in B\}$$

  is an ideal of B.

  - Catino, Colazzo and Stefanelli (2019), *Skew left braces with non-trivial annihilator*, J. Algebra App., 18(2).

- given I, J ideals of B with $J \subseteq I$, I/J is a central factor if $I/J \subseteq \zeta(B/J)$.

Central nilpotency

$$\text{Central nilpotency} \to \text{Strong nilpotency} \to \begin{cases} \text{Right nilpotency} \\ \\ \text{Left nilpotency} \end{cases}$$

Centrally nilpotent braces

Let B be a brace:

- $\zeta(B) = \mathrm{Soc}(B) \cap \mathrm{Fix}(B) =$

  $$= \{a \in B : b + a = a + b = ab = ba,\ \forall b \in B\}$$

  is an ideal of B.

  - Catino, Colazzo and Stefanelli (2019), *Skew left braces with non-trivial annihilator*, J. Algebra App., 18(2).

- given I, J ideals of B with $J \subseteq I$, I/J is a central factor if $I/J \subseteq \zeta(B/J)$.

Definition

A brace B is defined to be centrally nilpotent if it has a central series, that is

$$I_0 = 0 \subseteq I_1 \subseteq \ldots \subseteq I_n = B$$

such that I_k/I_{k-1} is a central factor for every $1 \leq k \leq n$.

- M. Bonatto and P. Jedlička (2022), *Central nilpotency of skew braces*, J. Algebra App., online.

Centrally nilpotent braces are soluble.

Definition

A brace B is defined to be centrally nilpotent if it has a central series, that is

$$I_0 = 0 \subseteq I_1 \subseteq \ldots \subseteq I_n = B$$

such that I_k/I_{k-1} is a central factor for every $1 \leq k \leq n$.

- M. Bonatto and P. Jedlička (2022), *Central nilpotency of skew braces*, J. Algebra App., online.

Centrally nilpotent braces are soluble.

Central series

Given B a brace, we can define canonical central series

1) $\zeta_0(B) = 0 \quad \zeta_{n+1}(B)$ is the ideal such that $\zeta_{n+1}(B)/\zeta_n(B) = \zeta(B/\zeta_n(B))$;

2) $\Gamma_1(B) = B; \quad \Gamma_{n+1}(B) = \langle \Gamma_n(B) * B, B * \Gamma_n(B), [\Gamma_n(B), B]_+ \rangle_+$;

3) $\Gamma_{[1]}(B) = B$;
$\Gamma_{[n]}(B) = \Big\langle \Gamma_{[i]}(B) * \Gamma_{[n-i]}, \Gamma_{[n-i]} * \Gamma_{[i]}(B), [\Gamma_{[i]}(B), \Gamma_{[n-i]}(B)]_+ :$
$: 1 \leq i \leq n-1 \rangle_+$

Theorem (BJ22, JVV23)

Let B be a brace. The following statements are equivalent

- *B is centrally nilpotent.*
- *There exists $n \in \mathbb{N}$ such that $\zeta_n(B) = B$.*
- *There exists $n \in \mathbb{N}$ such that $\Gamma_{n+1}(B) = 0$.*
- *There exists $c \in \mathbb{N}$ such that $\Gamma_{[c]}(B) = 0$.*

- E. Jespers, A. Van Antwerpen and L. Vendramin (2023), *Central nilpotency of skew braces*, Comm. Contemp. Math., online.

Question

1. Does the lower and the upper central series play an analogous role that in group theory? Is there a centrally nilpotent class?
2. For every $n \in \mathbb{N}$, does it follow $\Gamma_{n+1}(B) = [\Gamma_n(B), B]_B$?
3. Given I, J ideals of a brace B, what is the relation between $[I, J]_B$ and $\langle I * J, J * I, [I, J]_+\rangle_+$?

Theorem

Let B be a brace and let I, J be ideals of B. Then,

1. $I * J + J * I + [I, J]_+$ *is a left ideal.*
2. $\langle I * J + J * I + [I, J]_+\rangle = \langle [I, J]_+, [I, J]_\cdot, \{ij - (i + j) : i \in I,\ j \in J\}\rangle_+$. *Therefore,* $[I, J]_B = \langle I * J + J * I + [I, J]_+\rangle$.

Question

1. Does the lower and the upper central series play an analogous role that in group theory? Is there a centrally nilpotent class?
2. For every $n \in \mathbb{N}$, does it follow $\Gamma_{n+1}(B) = [\Gamma_n(B), B]_B$?
3. Given I, J ideals of a brace B, what is the relation between $[I, J]_B$ and $\langle I * J, J * I, [I, J]_+ \rangle_+$?

Theorem

Let B be a brace and let I, J be ideals of B. Then,

1. $I * J + J * I + [I, J]_+$ *is a left ideal.*
2. $\langle I * J + J * I + [I, J]_+ \rangle = \langle [I, J]_+, [I, J]_\cdot, \{ij - (i + j) : i \in I,\ j \in J\} \rangle_+$. *Therefore,* $[I, J]_B = \langle I * J + J * I + [I, J]_+ \rangle$.

Corollary

Let B be a brace. $\Gamma_1(B) = B$ and

$$\Gamma_n(B) = \Gamma_{n-1}(B) * B + B * \Gamma_{n-1} + [\Gamma_{n-1}(B), B]_+ = [\Gamma_{n-1}(B), B]_B$$

for every $n \in \mathbb{N}$.

Corollary

Let $0 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n = B$ a central series of a centrally nilpotent brace B. The following hold

1. *For every $0 \leq j \leq n$, $I_j \subseteq \zeta_j(B)$. In particular, $\zeta_n(B) = B$.*
2. *For every $1 \leq j \leq n+1$, $\Gamma_j(B) \subseteq I_{n-j+1}$. In particular, $\Gamma_{n+1}(B) = 0$.*

Then, the centrally nilpotent class of B is equal to the length of the upper and lower central series of B.

A Fitting ideal

Definition

Let I be an ideal of a brace B. We say that I is centrally nilpotent, if so it is as brace.

We say that I is *centrally nilpotent respect to* B if there exists a sequence of ideals in B

$$I_0 = 0 \subseteq I_1 \subseteq \ldots \subseteq I_n = I$$

such that $I_k/I_{k-1} \subseteq \zeta(I/I_{k-1})$ for every $1 \leq k \leq n$.

Theorem

Let B be a finite brace. Then, B is centrally nilpotent if, and only if, every Sylow p-subgroup is a centrally nilpotent ideal.

A Fitting ideal

Definition

Let I be an ideal of a brace B. We say that I is centrally nilpotent, if so it is as brace.
We say that I is *centrally nilpotent respect to* B if there exists a sequence of ideals in B

$$I_0 = 0 \subseteq I_1 \subseteq \ldots \subseteq I_n = I$$

such that $I_k/I_{k-1} \subseteq \zeta(I/I_{k-1})$ for every $1 \leq k \leq n$.

Theorem

Let B be a finite brace. Then, B is centrally nilpotent if, and only if, every Sylow p-subgroup is a centrally nilpotent ideal.

Theorem

Let I, J be ideals respect to B. Then, $IJ = I + J$ is also centrally nilpotent respect to B.

Definition

We define $\mathrm{Fit}(B)$, the Fitting ideal of a brace, as the ideal generated by all centrally nilpotent ideals respect to B.

Theorem

Assume that B is a brace satisfying the maximal condition on ideals. Then, $\mathrm{Fit}(B)$ is centrally nilpotent respect to B.

Theorem

$\mathrm{Fit}(B)$ *is the intersection of the centralisers in B of all chief factors in B.*

Theorem

If B is a soluble brace, then $\mathrm{C}_B(\mathrm{Fit}(B)) \subseteq \mathrm{Fit}(B)$.

Here, given an ideal I of a brace B, $\mathrm{C}_B(I)$ is the greatest ideal such that $[I, \mathrm{C}_B(I)]_B = 0$.