Advances on Quillen's conjecture

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Let X be a (finite) poset. The *reduced homology of X with coefficients in R* is:

$$C_m(X,R) = \text{free } R\text{-module on chains } x_0 < \ldots < x_m,\ m \geq -1,$$

$$d_m(x_0 < \ldots < x_m) = \sum_i (-1)^i (x_0 < \ldots < \hat{x}_i < \ldots < x_m),$$

$$\tilde{H}_m(X,R) = \mathrm{Ker}(d_m)/\mathrm{Im}(d_{m+1}).$$

- X is R-acyclic if $\tilde{H}_*(X,R) = 0$.
- $\mathcal{K}(X) =$ order-complex of X, then $\tilde{H}_*(X,R) = \tilde{H}_*(\mathcal{K}(X),R)$.
- Topology of X = topology of $\mathcal{K}(X)$.
- Homotopy type of X = homotopy type of $\mathcal{K}(X)$.

Setting

Let G be a finite group and p a prime number.

The Quillen poset is:

$$\mathcal{A}_p(G) = \{A \leq G \ : \ A \text{ is a non-trivial elementary abelian } p\text{-group}\},$$

- A is an *elementary abelian p-group* if $A \cong C_p \times \ldots \times C_p \cong \mathbb{Z}/p\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p\mathbb{Z}$.
- $\mathcal{A}_p(G)$ is a finite poset with the order induced by the inclusion.
- G acts on $\mathcal{A}_p(G)$ by conjugation.

General goal. Establish connections between properties of G and combinatorial/topological properties of $\mathcal{A}_p(G)$.

Why do we study p-group complexes?

1. (D. Quillen, '71) The Atiyah-Swan conjecture holds. Namely,

   Krull dimension of $H^*(G,k) = p$-rank of $G = 1 + \dim \mathcal{A}_p(G)$.

2. (K. Brown, '94) $H^*_G(\mathcal{A}_p(G), p) \cong H^*(G,p)$.

3. (Quillen, '78) $\mathcal{A}_p(G)$ is disconnected if and only if G has a strongly p-embedded subgroup. These groups are crucial in the **CFSG**!

4. (Quillen, '78) $O_p(G) =$ largest normal p-subgroup of G.
   If $O_p(G) \neq 1$ then $\mathcal{A}_p(G)$ is contractible.

Quillen's conjecture

If $O_p(G) = 1$ then $\mathcal{A}_p(G)$ is not contractible.

(Strong) Quillen's conjecture

If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.

$$\textbf{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0.$$

Quillen proved the following cases of (H-QC):

1. G is a group of Lie type in characteristic p;
2. p-rank of $G = m_p(G) \leq 2$;
3. G is solvable. Moreover, if $O_p(G) = 1$ then G satisfies $(\mathcal{QD})_p$.

- H satisfies $(\mathcal{QD})_p$ if $\mathcal{A}_p(H)$ has non-zero homology in top-degree:

$$\tilde{H}_{m_p(H)-1}(\mathcal{A}_p(H), \mathbb{Q}) \neq 0.$$

Theorem. If G is p-solvable and $O_p(G) = 1$ then G satisfies $(\mathcal{QD})_p$, and hence (H-QC).

Aschbacher-Smith Theorem. (H-QC) holds for G if $p > 5$ and:

(HU) If $L \cong \mathrm{PSU}_n(q)$, $p \mid q+1, q$ odd, is a component of G, then p-extensions of $\mathrm{PSU}_m(q^e)$ satisfy $(\mathcal{QD})_p$ $\forall m \leq n, e \in \mathbb{Z}$.

- A p-extension of L is a split-extension of L by some $B \in \mathcal{A}_p(\mathrm{Out}(L)) \cup \{1\}$:

$$1 \longrightarrow L \longrightarrow LB \longrightarrow B \longrightarrow 1.$$

1. Under a minimal counterexample G to (H-QC), if p is odd and G does not contain the following components:

$$L = \mathrm{Sz}(2^5)(p = 5),\ \mathrm{PSL}_2(2^3)(p = 3),\ \mathrm{PSU}_3(2^3)(p = 3),$$

then "several reductions" are possible.

2. E.g. every component L has a p-extension failing $(\mathcal{QD})_p$.
3. Short list of simple groups such that some p-extension fails $(\mathcal{QD})_p$ for p odd. $\mathrm{PSU}_n(q), p \mid q + 1$, are in this list!

On Quillen's conjecture: new results

- Alternative methods allow us to eliminate the problematic components

$$L = \mathrm{Sz}(2^5)(p=5),\ \mathrm{PSL}_2(2^3)(p=3),\ \mathrm{PSU}_3(2^3)(p=3).$$

Theorem

Aschbacher-Smith Theorem extends to $p = 5$.

Extension to $p = 3$: be careful with components $L = \mathrm{Ree}(3^a)$.

Theorem

Aschbacher-Smith Theorem extends to $p = 3$ (so to every odd prime).

Idea. Replace strongly **CFSG**-dependent steps with combinatorial arguments.

Theorem

If $p = 2$ and G is a minimal counterexample to (H-QC), then:

1. $O_{2'}(G) = 1$,
2. every component L of G has non-trivial 2-extension $LB \leq G$,
3. every component L of G has a 2-extension in G failing $(\mathcal{QD})_2$,
4. G has a component L of Lie type such that $\mathrm{char}(L) \neq 2, 3$ or

$$L \cong \mathrm{PSL}_n(2^a)(n \geq 3), D_n(2^a)(n \geq 4), \text{ or } E_6(2^a).$$

On Quillen's conjecture: more recent results for $p = 2$

By the previous theorem, if G is a minimal counterexample for (H-QC) and $p = 2$, then every simple component of G has a 2-extension failing $(\mathcal{QD})_2$:

$$LB \leq G, \; L \text{ a simple component and } LB \text{ a 2-extension such that}$$

$$\text{not-}(\mathcal{QD})_2 \qquad H_{m_2(LB)-1}(\mathcal{A}_2(LB), \mathbb{Q}) = 0.$$

Problem. Classify simple groups L satisfying the following condition:

(E-$(\mathcal{QD})$) Every 2-extension of L satisfies $(\mathcal{QD})_2$.

Theorem

Let L be a simple group of exceptional Lie type in odd characteristic. If L fails (E-$(\mathcal{QD})$), then it is one of the following groups:

$$^3D_4(9), F_4(3), F_4(9), G_2(3), G_2(9), {}^2G_2(3)', E_8(3), E_8(9).$$

Idea of the proof

1. Establish (E-($\mathcal{QD}$)) for the low-rank groups $\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3$. Use counting arguments (conjugacy classes of 2-subgroups, involutions, field and graph automorphisms).
2. For L an exceptional group, we look for maximal subgroups H of the 2-extensions LB such that

$$H \leq LB, \quad m_2(H) = m_2(LB), \quad \text{and}$$

$$0 \neq H_{m_2(H)-1}(\mathcal{A}_2(H), \mathbb{Q}) \subseteq H_{m_2(LB)-1}(\mathcal{A}_2(LB), \mathbb{Q}).$$

3. If H is parabolic, it has a solvable subgroup K with $m_2(H) = m_2(K)$ and $O_2(K) = 1$, so we are done by Quillen's result.
4. Otherwise, look for $H = H_1 \times H_2$, where the H_i satisfy $(\mathcal{QD})_2$ and

$$H_{m_2(H)-1}(\mathcal{A}_2(H), \mathbb{Q}) = H_{m_2(H_1)-1}(\mathcal{A}_2(H_1), \mathbb{Q}) \otimes H_{m_2(H_2)-1}(\mathcal{A}_2(H_2), \mathbb{Q}).$$

Consequence on Quillen's conjecture for $p = 2$

Corollary

Let G be a minimal counterexample to (H-QC) for $p = 2$. Then G contains a component of Lie type in characteristic $r \neq 3$.
Moreover, every such component fails (E-($\mathcal{QD}$)) and belongs to one of the following families:

$$\mathrm{PSL}_n(2^a)(n \geq 3), D_n(2^a)(n \geq 4), E_6(2^a),$$

$$\mathrm{PSL}_n^{\pm}(q)(n \geq 4), \Omega_{2n+1}(q)(n \geq 2), \mathrm{PSp}_{2n}(q)(n \geq 3), D_n^{\pm}(q)(n \geq 4),$$

where $q = r^b$ and $r > 3$.

What to do next? Study (E-($\mathcal{QD}$)) for the classical groups. Partial results for $\Omega_{2n+1}(q), \mathrm{PSp}_{2n}(q)$ and some of the $D_n^{\pm}(q)$. But I'd try a different argument to eliminate them...

Thanks for your attention!