ZESTING LINKS INVARIANTS

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GROUPS, RINGS, AND THE YANG-BAXTER EQUATION

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WHY MTC'S ? → MODULAR TENSOR CATEGORIES

- THEY APPEAR IN MANY AREAS OF MATH & HAVE APPLICATIONS IN PHYSICS.
- NATURAL HOSTS OF QUANTUM SYMMETRIES.

WHAT ABOUT THEM ?

- "YOUNG" THEORY...

WE NEED MORE EXAMPLES!

SOME PROBLEMS I LIKE TO THINK ABOUT:

- CLASSIFICATION
- PROPERTIES / INVARIANTS
- CONSTRUCTIONS

ZESTING ...   History & Motivation

- **2014:**
  Classification of MTC's of FPdim = 36
  Rank 10 fusion ring similar but it not the same as $SU(3)_3$ → "Twist" fusion rules

- **2016:**
  Minimal closures of super-modular categories given one MME → there's exactly 16
  [DMNO], [KLW], [B+], [JF.R]

  $PSU(2)_{4m+2} \subseteq SU(2)_{4m+2}$ → 8 of them @ ZEST.
  #
  $\subseteq SO(2m+1)_2$ → 8 of them ⊃ ZEST

- Categorification problems: - Grossman, Izumi (2018)
  Non modular data [BRW]
- Fusion rings (2019, 2020) Liu, Palcoux, Ren

Definition ... By example (of braided fusion category)

$\mathcal{C} = \mathrm{Rep}(G)$ = finite dimensional representations of G over k ($k = \mathbb{C}$, char $k = 0$)

Given $V, W \in \mathrm{Rep}(G)$:

- $\mathrm{Hom}_{\mathcal{C}}(V, W)$ = intertwiners $\rightsquigarrow$ k-v.s.
- $V \otimes W \in \mathrm{Rep}(G)$ $\rightsquigarrow$ $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$
- $k \in \mathrm{Rep}(G)$ $\rightsquigarrow$ $g \cdot 1 = 1$
- $V^* \in \mathrm{Rep}(G)$, $T \in V^*$ $\rightsquigarrow$ $(g \cdot T)(v) = T(g^{-1} \cdot v)$

  $V^* = \mathrm{Lin}(V, k)$
- $\tau : V \otimes W \to W \otimes V \in \mathrm{Hom}_{\mathcal{C}}(V \otimes W, W \otimes V)$

  $v \otimes w \mapsto w \otimes v$

Notation & Definitions: $\Bbbk = \mathbb{C}$.

DEF: We say that $\mathcal{C}$ is a **MODULAR TENSOR CATEGORY** (MTC) if:

- Abelian $\mathbb{C}$-linear: $\oplus$, ker, coker, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\mathbb{C}$-v.s.
- Monoidal: $(\otimes, \alpha, \underset{\in \mathcal{C}}{\mathbb{1}}, \ell, r)$ + pentagon + $\triangle$ ax. — simple
- Rigid $\forall X \in \mathcal{C}$: $(\underset{\in \mathcal{C}}{X^*}, \mathrm{ev}_X : X^* \otimes X \to \mathbb{1}, \mathrm{coev}_X : \mathbb{1} \to X \otimes X^*)$ + 2 zig-zag ax.
- Semisimple
- Finite: fin. many simple obj. (up to isom.)
- Braided: $\sigma_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ + 2 hexagon's ax.
- Ribbon: $\theta_X : X \xrightarrow{\sim} X$ twist $\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ \sigma_{Y,X} \circ \sigma_{X,Y}$
- Non-degenerate: $Z_2(\mathcal{C}) = \{X \in \mathcal{C} \,/\, \sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X \otimes Y} \ \forall Y\} = \mathrm{VEC}$

NOTATION: $\mathcal{C}$ fusion cat.

- $\mathrm{Irr}(\mathcal{C}) = \{ X \in \mathcal{C} \text{ simple} \} / \sim$

  $\mathrm{Rank}(\mathcal{C}) = |\mathrm{Irr}(\mathcal{C})| < \infty$

- $\mathrm{Inv}(\mathcal{C}) = \{ X \in \mathcal{C} \text{ invertible} \}$ $\quad X \otimes X^* \simeq \mathbb{1} \simeq X^* \otimes X$

  $G(\mathcal{C}) = \mathrm{Inv}(\mathcal{C}) / \sim$

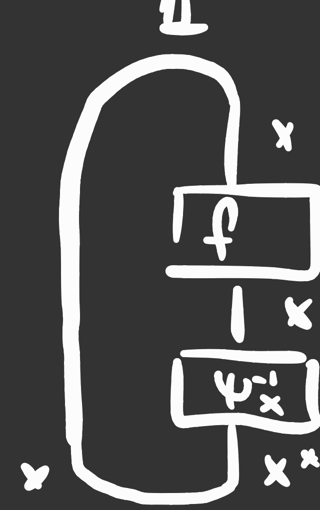
- **Fusion rules**: $X, Y \in \mathrm{Irr}(\mathcal{C})$

$$X \otimes Y = \bigoplus_{Z \in \mathrm{Irr}(\mathcal{C})} N_{X,Y}^{Z} \, Z \qquad N_{X,Y}^{Z} \in \mathbb{N}_0$$

Modular Data

$\mathcal{C}$ premodular

- **Trace**: $f \in \mathrm{End}(X) \rightsquigarrow \mathrm{Tr}(f) =$



pivotal struct.

- **T-matrix**: $\theta_X : X \xrightarrow{\sim} X$ twist.

$X \in \mathrm{Irr}(\mathcal{C}) \longrightarrow \theta_X \in \mathrm{End}(X) \simeq \mathbb{C}\, id_X$

$\theta_X \in \mathbb{C}^\times \rightsquigarrow T_{X,Y} = \delta_{X,Y}\theta_X$

- **S-matrix**:

$$S_{X,Y} = \mathrm{Tr}(\sigma_{Y,X} \circ \sigma_{X,Y}) = \frac{1}{D}$$

Hopf link

$X, Y \in \mathrm{Irr}(\mathcal{C})$

$\mathcal{C}$ Modular $\longleftrightarrow$ $\det S \neq 0$

EXAMPLES:

1) POINTED: abelian G finite group, q non deg. quadr. form

↳ $\mathrm{Irr}(\mathcal{C}) = G(\mathcal{C})$

Vec_G^q = fin. dim. G-graded v.s.

2) DOUBLES: G finite group, $\omega \in H^3(G, \mathbb{C}^*)$

$$\mathrm{Rep}(D^\omega G) = Z(\mathrm{Vec}_G^\omega)$$

↳ Drinfeld center

3) QUANTUM GROUPS: $\mathfrak{g}$ simple Lie Alg $\quad q = e^{\pi i/\ell}$

semisimplified

$$U\mathfrak{g} \longrightarrow U_q\mathfrak{g} \longrightarrow \overline{\mathrm{Rep}(U_q\mathfrak{g})}$$

MIGNARD - SCHAUENBURG EXAMPLE

CONJECTURE: THE MODULAR DATA (S,T) DETERMINES THE MODULAR CATEGORY.

↳ FALSE!

EXAMPLE: LET p, q BE ODD PRIMES S.T. $p \mid q-1$.

CONSIDER $G = \mathbb{Z}_q \rtimes_n \mathbb{Z}_p$ WITH $n^p \equiv 1 \mod q$.

LET $\mathcal{C} = \mathrm{Rep}(D^{\omega^v} G)$ → p NON EQUIV. MTC'S.

$\omega \in H^3(G, \mathbb{C}^*) \cong \mathbb{Z}_p = \langle \omega \rangle$

$v = 0, \ldots, p-1$

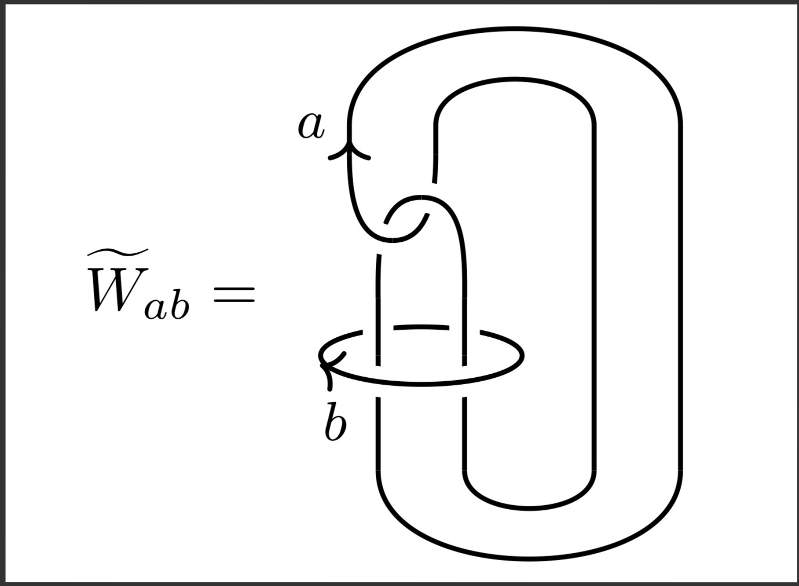
BUT THERE IS AT MOST 3 SETS OF MODULAR DATA

INVARIANTS BEYOND MODULAR DATA

- **W-MATRIX**

[BONDERSON, DELANEY, GALINDO, ROWELL, TRAN, WANG]

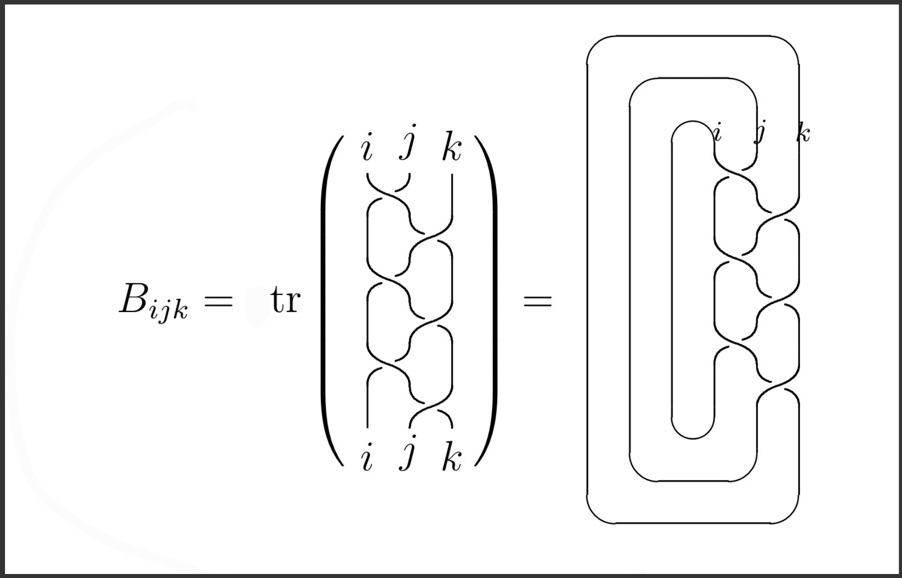
WHITEHEAD LINK



- **B-TORSOR:**

[KULKARNI, MIGNARD, SCHAUENBURG]

BORROMEAN LINK



THE ZESTING CONSTRUCTION

OVERVIEW:

INPUT: $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ G-GRADED, PREMODULAR CATEGORY

STEP 1: ZEST FUSION RULES: $\tilde{\otimes}, \tilde{\alpha}$

STEP 2: ZEST BRAIDING: $\tilde{\sigma}$

STEP 3: ZEST RIBBON: $\tilde{\theta}$

OUTPUT: $\tilde{\mathcal{C}} = \bigoplus_{g \in G} \mathcal{C}_g$ PREMODULAR CAT.

G-GRADINGS ON FUSION CATEGORIES

Let $\mathcal{C}$ be a fusion category.

We say that $\mathcal{C}$ is **G-GRADED** (faithful) if it decomposes

as $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$

- $\mathcal{C}_g$ full ab. subc., $\mathcal{C}_g \neq 0$
- $\mathcal{C}_g \times \mathcal{C}_h \xrightarrow{\otimes} \mathcal{C}_{gh}$

REMARKS: • $\mathcal{C}_e$ fusion subc.

- [Gelaki, Nikshych] $U(\mathcal{C})$ = universal grading group faithfully grades $\mathcal{C}$.

$$\mathcal{C}_e = \mathcal{C}_{ad} = \langle X \otimes X^* \,/\, X \in \mathrm{Irr}\,\mathcal{C} \rangle$$

If $\mathcal{C}$ is an MTC: $U(\mathcal{C}) \simeq G(\mathcal{C})$

ASSOCIATIVE TWISTING

SET UP: $\mathcal{C} = \bigoplus_{a \in A} \mathcal{C}_a$ A - GRADED BRAIDED fusion CAT.

ABELIAN

GOAL: Modify in an EASY way its fusion rules to get a new fusion category

$X_a \in \mathcal{C}_a,\ Y_b \in \mathcal{C}_b \rightsquigarrow X_a \tilde{\otimes} Y_b = X_a \otimes Y_b \otimes \lambda(a,b)$

?

$\operatorname{INV}(\mathcal{C}_e)$

ASSUME: $\lambda(e,a) = \mathbb{1} = \lambda(a,e)$

WHAT ABOUT THE ASSOCIATIVITY?

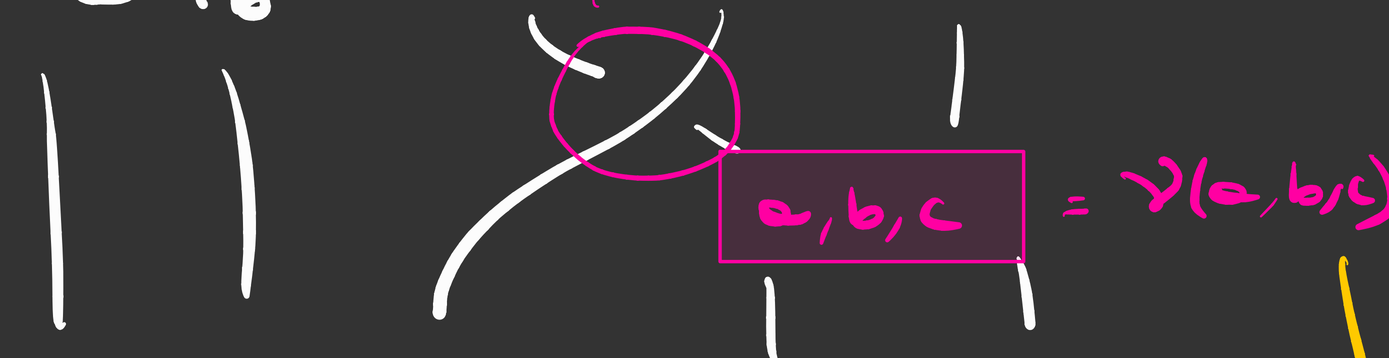
(for simplicity, assume $\mathcal{C}$ strict).

$$\tilde{\alpha}_{X_a,Y_b,Z_c} : (X_a \tilde{\otimes} Y_b) \tilde{\otimes} Z_c \xrightarrow{\sim} X_a \tilde{\otimes} (Y_b \tilde{\otimes} Z_c)$$

Braid

$$(X_a \tilde{\otimes} Y_b) \tilde{\otimes} Z_c = X_a Y_b \lambda(a,b) Z_c \lambda(ab,c)$$

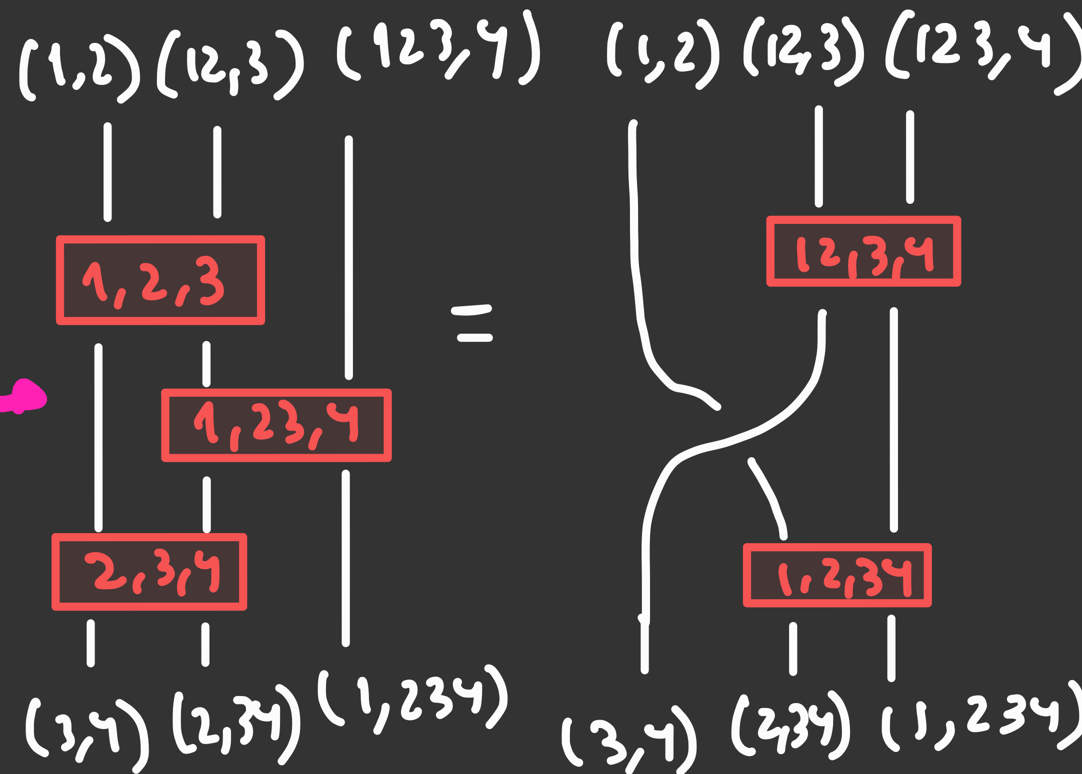
$$\Big\downarrow \tilde{\alpha}_{X_a,Y_b,Z_c}$$



$$X_a \tilde{\otimes} (Y_b \tilde{\otimes} Z_c) = X_a Y_b Z_c \lambda(b,c) \lambda(a,bc)$$

2-cocycle cond. on $\lambda(a,b)$

Axiom for $(\tilde{\otimes}, \tilde{\alpha})$



Obstruction in $H^4(A, \mathbb{C}^\times)$

REMARKS :

- RANK , FPdim $\begin{cases} \text{OBJ.} \\ \text{CAT.} \end{cases}$ fixed
- GRADING is fixed
- COHOMOLOGICAL OBSTRUCTION
- ASSOCIATIVE ZESTINGS form A TORSOR OVER $H^3(A, k^\times)$
- EXTENSION THEORY END

BRAIDED ZESTING

SET UP:

- $\mathcal{C} = \bigoplus_{a \in A} \mathcal{C}_a$ A-graded braided fusion cat.
- $\tilde{\mathcal{C}} = \mathcal{C}^{\lambda} \rightarrow$ associative zesting of $\mathcal{C}$.

Goal: Modify in an easy way the braid of $\mathcal{C}$ to get a braid in $\tilde{\mathcal{C}}$.

?

$$\tilde{c}_{X_a, Y_b} : X_a \tilde{\otimes} Y_b \xrightarrow{\sim} Y_b \tilde{\otimes} X_a$$

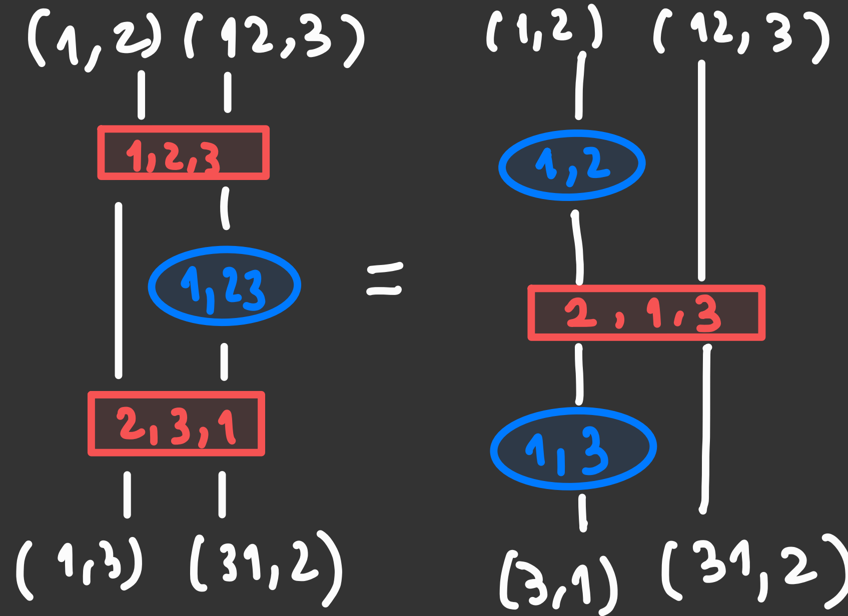
$X_a \tilde{\otimes} Y_b = X_a \quad Y_b \quad \lambda(a,b)$

$\tilde{c}_{X_a,Y_b} \downarrow \sim$

$Y_b \tilde{\otimes} X_a = Y_b \quad X_a \quad \lambda(b,a)$

⬡'s Axioms

for $(\tilde{\otimes}, \tilde{\alpha}, \tilde{\sigma})$ $\longleftrightarrow$



+ Another similar diagram with inverses & an extra scalar factor

Remarks:

- Cohomological obstructions
- Braided zesting (often) form over bicharact. of A
- Braided extension theory

PROPERTIES :

- **MÜGER CENTER :** CHANGES (CAN) A LOT

BUT :

$\mathcal{C}$ MTC , $\mathcal{C}_{pt} \overset{\langle G(\mathcal{C}) \rangle}{=}$ symm. Any $G(\mathcal{C})$ - GRADING

it is modular

- **MODULAR DATA :** $(\tilde{S}, \tilde{T})$ in terms of (S, T)

AND GRADING DATA

- **BRAID IMAGE :** $\mathcal{C}$ PREMODULAR CAT.

$\mathcal{C}$ satisfies PROP. F $\leftrightarrow$ $\tilde{\mathcal{C}}$ has PROP. F

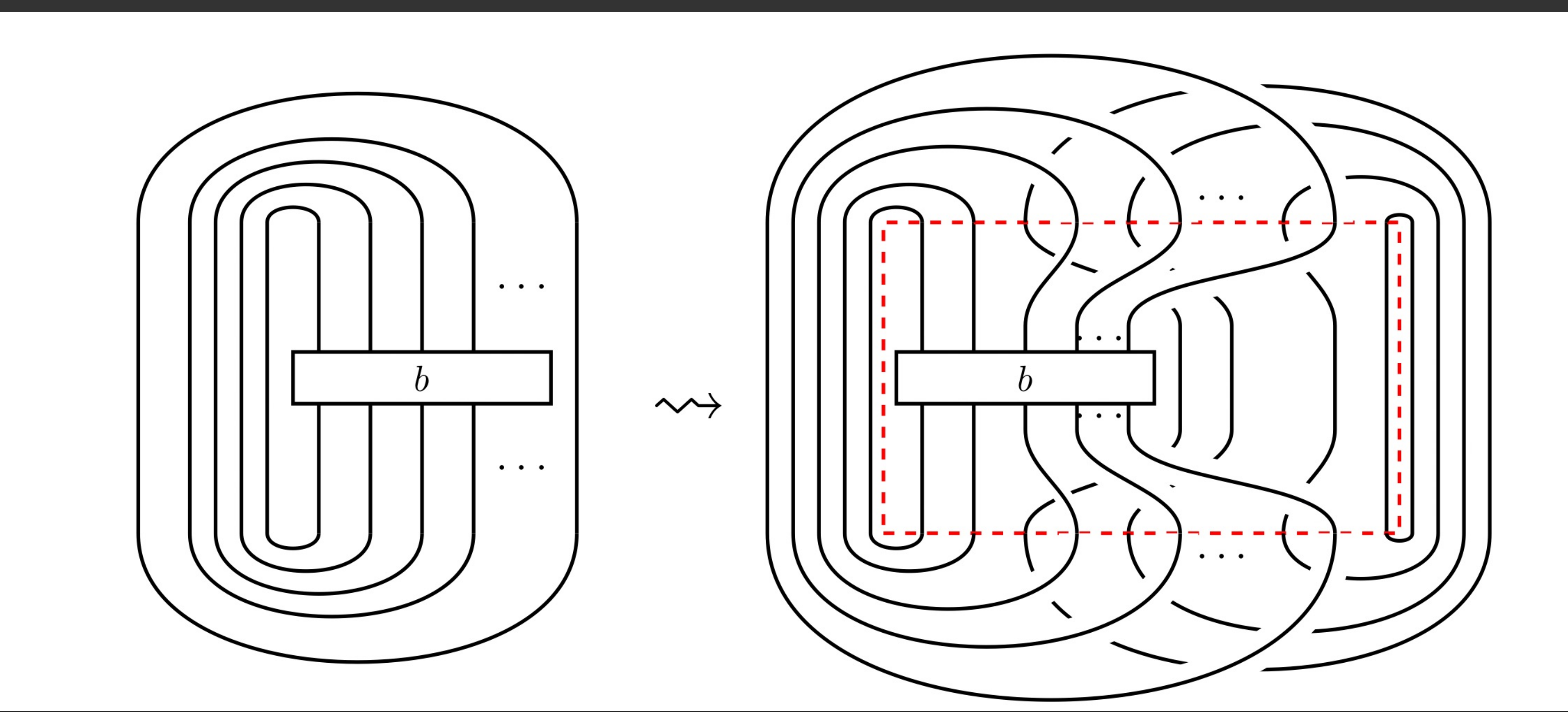
LINK INVARIANTS UNDER ZESTING

THEOR.: [DELANEY, KIM, P.]

LET $\mathcal{C}^{(\lambda,t,f)}$ BE RIBBON ZESTING OF $\mathcal{C}$ &

LET L BE A FRAMED n-COMPONENT LINK INVARIANT.

THEN $\tilde{L} = L^{(\lambda,t,f)}_{i_1,\dots,i_n} = \overbrace{\boxed{h_{(\lambda,t,f,i_1,\dots,i_n)}}}^{\in\mathbb{C}} L_{i_1,\dots,i_n}$



MIGNARD-SCHAUENBURG ISOTOPES

DEF.: [DELANEY] A SET OF INEQUIVALENT MTC'S ARE CALLED **MODULAR ISOTOPES** IF THEY SHARE (S, T)

MIGNARD-SCHAUENBURG EXAMPLES:

$\mathrm{REP}(D^{\omega}G)$ WITH $G = \mathbb{Z}_q \rtimes \mathbb{Z}_p$ $p \mid q-1$ ODD PRIMES

THEOREM: [DKP] $Z(\mathrm{VEC}_G^{\omega^u})$ ARE RELATED BY ZESTING

THEOREM:
- [BDGRTW] (W, T) → DISTINGUISH
- [KMS] (B, T) → THEM.

QUESTIONS

COMMENTS

THANK

YOU!