About Non-degenerated Involutive solutions of the Yang-Baxter Equation

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Groups, Rings and the Yang-Baxter equation, Blankerberge

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Formulation

Definition

A set-theoretical solution of the Yang-Baxter equation is a pair (X, S) where X is a non-empty set and S a map from $X \times X$ to itself so that

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All solutions considered will be finite non-degenerate and involutive.

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The structure group of a solution (X, S)

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Theorem ([9, 2.14])

Given a solution (X, S), its structure group $G(X, S)$ is solvable.

Structures

Non-degenerate involutive solutions are related with several strucures, such as:

1. Left Braces
2. Cycle Sets
3. Garside Monoids
4. Linear groups
5. l-groups

Decomposable Solutions

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2. An invariant subset $Y \subset X$ is said to be non-degenerate if $(Y, S|_{Y \times Y})$ is a non-degenerate involutive solution.
3. (X, S) is said to be decomposable if it is a union of two nonempty disjoint non-degenerate invariant subsets. Otherwise, (X, S) is said to be indecomposable.

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Theorem ([6, A])

Let (X,S) be a solution with $|X| > 1$. If the order of T and the cardinality of X are coprime, then (X,S) is decomposable.

Retractable Solutions

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This relation induces a new solution $\left(\overline{X}, \overline{S}\right)$ where $\overline{S}\left(\overline{x}, \overline{y}\right) = \left(\overline{g_x\left(y\right)}, \overline{f_y\left(x\right)}\right)$, called retracted solution, also denoted by $\operatorname{Ret}(X, S)$.

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If there exists m so that $\mathrm{Ret}^m(X, S)$ has cardinality one, then (X, S) is called multipermutation solution of level m.

About Retraction

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Theorem ([4, 3.5])

Let (X, S) be a solution. If the permutation group $\mathscr{G}_{(X,S)}$ has an abelian normal Sylow p-subgroup, then (X, S) is retractable.

About Multipermuation

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It is false in the general case. However it holds for the abelian case, and if no power of a prime divides $|X|$. See [3, 1, 4]

About Multipermuation

Lemma ([11, 34])

Let (X, S) be a solution of a finite multipermutation level and $Y \subseteq X$ be such that $S(Y, Y) \subseteq (Y, Y)$, then $\left(Y, S_{|Y}\right)$ is also a solution with $mpl(Y, S') \leq mpl(X, S)$.

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Lemma ([11, 36])

Let (X, S) *be a finite multipermutation solution with* $|X| > 1$*. If for every* $x \in X$ *there is* $y \in X$ *such that* $S(x, y) = (y, x)$*, then the solution* (X, S) *is decomposable.*

About Imprimitivity

Theorem ([9, 2.12])

Let (X, S) be an indecomposable solution with $|X| = p$, a prime. Then (X, S) is isomorphic to the cyclic permutation solution $(\mathbb{Z}/p\mathbb{Z}, S_0)$, where $S_0(x, y) = (y - 1, x + 1)$.

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Theorem ([2, 3.1])

Let (X, S) be a finite "primitive solution" of the YBE with $|X| > 1$. Then $|X|$ is prime.

About Primes

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Given a solution (X, S) of square-free cardinality with $|X| = p_1 \cdot \dots \cdot p_k$, the following hold:

1. *$p_1, \dots, p_n$ are the only primes dividing the order of $\mathscr{G}_{(X,S)}$.*
2. *The Sylow p_i-subgroups of $\mathscr{G}_{(X,S)}$ are elementary abelian.*

About Primes

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Theorem (Sergio Camp, Raúl Sastriques)

Let (X,S) indecomposable solution. If $\mathscr{G}_{(X,S)}$ has an element of prime order q, then either:

- q divides $|X|$, or
- q divides some $p^n - 1$, with p prime and p^n dividing $|X|$.

Moreover, if $|X| = p^m$ and $p \neq q$, q has to divide $p^r - 1$, for some $r \leq m-1$.

About Simple Solutions

Definition ([4])

We say that a finite indecomposable solution (X, S) of the YBE has **primitive level k** if k is the biggest positive integer such that there exist solutions $(X, S) = (X_1, S_1), (X_2, S_2), ..., (X_k, S_k)$ and epimorphisms of solutions $p_i + 1$: $(X_i, S_i) \to (X_{i+1} S - i + 1)$with $|X_i| > |X_{i+1}| > 1$, for $1 \leq i \leq k - 1$, and (X_k, S_k) is primitive.

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A solution (X,S) is simple if $|X|>1$ and for every epimorphism $f:(X,S)\longrightarrow(Y,S')$ of solutions either f is an isomorphism or $|Y|=1$.

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Lemma ([4, 3.2])

Assume that (X,S) is a simple solution of the YBE. Then it is indecomposable if $|X|>2$ and it is irretractable if $|X|$ is not a prime number.

End

Thanks for your attention !

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