Quotient gradings and IYB groups

Yuval Ginosar and Ofir Schnabel

June 22, 2023

Groups, Rings and the Yang-Baxter equation

Definitions

- A grading of an algebra $\mathcal{A}$ by a group Γ is a vector space decomposition $\mathcal{A} = \bigoplus_{\gamma\in\Gamma} \mathcal{A}_{\gamma}$ such that $\mathcal{A}_{\gamma_1} \cdot \mathcal{A}_{\gamma_2} \subseteq \mathcal{A}_{\gamma_1 \cdot \gamma_2}$ for every $\gamma_1, \gamma_2 \in \Gamma$.

Definitions

- A grading of an algebra $\mathcal{A}$ by a group Γ is a vector space decomposition $\mathcal{A} = \bigoplus_{\gamma\in\Gamma} \mathcal{A}_\gamma$ such that $\mathcal{A}_{\gamma_1} \cdot \mathcal{A}_{\gamma_2} \subseteq \mathcal{A}_{\gamma_1\cdot\gamma_2}$ for every $\gamma_1, \gamma_2 \in \Gamma$.
- The subalgebra $\mathcal{A}_e$ is called the base algebra of the grading, for e the identity element of Γ.

Examples

- The natural $\mathbb{Z}$-grading of a polynomial ring $\mathcal{A} = \mathbb{C}[x]$. Here $\mathcal{A}_i = \text{span}\{x^i\}$.

Examples

- The natural $\mathbb{Z}$-grading of a polynomial ring $\mathcal{A} = \mathbb{C}[x]$. Here $\mathcal{A}_i = \text{span}\{x^i\}$.
- The natural G-grading of the group algebra $\mathcal{A} = \mathbb{C}G$. Here $\mathcal{A}_g = \text{span}\{g\}$.

Examples

- The natural $\mathbb{Z}$-grading of a polynomial ring $\mathcal{A} = \mathbb{C}[x]$. Here $\mathcal{A}_i = \text{span}\{x^i\}$.
- The natural G-grading of the group algebra $\mathcal{A} = \mathbb{C}G$. Here $\mathcal{A}_g = \text{span}\{g\}$.
- More generally, a twisted group algebra $\mathbb{C}^\alpha G$ is an associative algebra with basis $\{u_g\}_{g \in G}$, where

$$u_{g_1} u_{g_2} = \alpha(g_1, g_2) u_{g_1 g_2},$$

for $\alpha \in Z^2(G, \mathbb{C}^*)$.

Examples

- The natural $\mathbb{Z}$-grading of a polynomial ring $\mathcal{A} = \mathbb{C}[x]$. Here $\mathcal{A}_i = \text{span}\{x^i\}$.
- The natural G-grading of the group algebra $\mathcal{A} = \mathbb{C}G$. Here $\mathcal{A}_g = \text{span}\{g\}$.
- More generally, a twisted group algebra $\mathbb{C}^\alpha G$ is an associative algebra with basis $\{u_g\}_{g \in G}$, where

$$u_{g_1} u_{g_2} = \alpha(g_1, g_2) u_{g_1 g_2},$$

  for $\alpha \in Z^2(G, \mathbb{C}^*)$.
- $\mathcal{A} = \mathbb{C}^\alpha G$ is equipped with a natural twisted grading given by $\mathcal{A}_g = \text{span}\{u_g\}$.

Example

Consider the following $G = C_n \times C_n = \langle\sigma\rangle \times \langle\tau\rangle$-grading of $\mathcal{A} = M_n(\mathbb{C})$. For η_n be an n-th primitive root of unity, let $\mathcal{A}_{\sigma^i\tau^j}$ =span$_{\mathbb{C}}(u_\sigma^i u_\tau^j)$, where

$$u_\sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad u_\tau = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \eta_n & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \eta_n^{n-1} \end{pmatrix}$$

Note that $u_\tau u_\sigma = \eta_n u_\sigma u_\tau$.

Example

Consider the following $G = C_n \times C_n = \langle\sigma\rangle \times \langle\tau\rangle$-grading of $\mathcal{A} = M_n(\mathbb{C})$. For η_n be an n-th primitive root of unity, let $\mathcal{A}_{\sigma^i\tau^j}$ =span$_{\mathbb{C}}(u_\sigma^i u_\tau^j)$, where

$$u_\sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad u_\tau = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \eta_n & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \eta_n^{n-1} \end{pmatrix}$$

Note that $u_\tau u_\sigma = \eta_n u_\sigma u_\tau$.

- This grading can be considered as a twisted group algebra $\mathbb{C}^\alpha G$ where $\alpha \in Z^2(G, \mathbb{C}^*)$ is defined by $u_{\sigma^i\tau^j} = u_\sigma^i u_\tau^j$.

Two types of gradings of $M_n(\mathbb{C})$

- We see that unlike group algebras, twisted group algebra $\mathbb{C}^\alpha G$ can be simple, that is matrix algebra $M_n(\mathbb{C})$ s.t $n^2 = |G|$. In this case we say that G is of central type and α (or $[\alpha]$) is nondegenerate. In those cases, $M_n(\mathbb{C})$ is equipped with the natural twisted grading.

Two types of gradings of $M_n(\mathbb{C})$

- We see that unlike group algebras, twisted group algebra $\mathbb{C}^\alpha G$ can be simple, that is matrix algebra $M_n(\mathbb{C})$ s.t $n^2 = |G|$. In this case we say that G is of central type and α (or $[\alpha]$) is nondegenerate. In those cases, $M_n(\mathbb{C})$ is equipped with the natural twisted grading.
- On the other hand, for any group H, an n-tuple $(h_1, h_2, \dots, h_n) \in H^n$ induces an elementary H-grading on $\mathcal{A} = M_n(\mathbb{C})$ by setting $\mathcal{A}_h = \text{span}\{E_{ij} | h = h_i h_j^{-1}\}$.

Two types of gradings of $M_n(\mathbb{C})$

- We see that unlike group algebras, twisted group algebra $\mathbb{C}^\alpha G$ can be simple, that is matrix algebra $M_n(\mathbb{C})$ s.t $n^2 = |G|$. In this case we say that G is of central type and α (or $[\alpha]$) is nondegenerate. In those cases, $M_n(\mathbb{C})$ is equipped with the natural twisted grading.
- On the other hand, for any group H, an n-tuple $(h_1, h_2, \ldots, h_n) \in H^n$ induces an elementary H-grading on $\mathcal{A} = M_n(\mathbb{C})$ by setting $\mathcal{A}_h = \text{span}\{E_{ij} | h = h_i h_j^{-1}\}$.
- A particular case which is called an elementary crossed product grading is where H is of order n and the n-tuple consists of the distinct elements in H.

Example of an elementary grading

Recall that $\mathcal{A}_h = \text{span}\{E_{ij} | h = h_i h_j^{-1}\}$ for $(h_1, h_2, \ldots, h_n) \in H^n$.

Example of an elementary grading

Recall that $\mathcal{A}_h = \text{span}\{E_{ij} | h = h_i h_j^{-1}\}$ for $(h_1, h_2, \ldots, h_n) \in H^n$.

- Let $\mathcal{A} = M_2(\mathbb{C})$ and let $G \cong C_2 = \langle \sigma \rangle$. Then the elementary grading determined by $(1, \sigma)$ is

$$\mathcal{A}_e = \text{span}\{E_{11}, E_{22}\}, \quad \mathcal{A}_\sigma = \text{span}\{E_{12}, E_{21}\}$$

Quotient gradings

- For a Γ-grading $\mathcal{A} = \bigoplus_{\gamma\in\Gamma} \mathcal{A}_\gamma$ and $N \lhd \Gamma$ there is a natural quotient Γ/N-grading given by

$$\mathcal{A} = \bigoplus_{\bar{\gamma}\in\Gamma/N} \mathcal{A}_{\bar{\gamma}},$$

where $\mathcal{A}_{\bar{\gamma}} := \bigoplus_{\gamma\in\bar{\gamma}} \mathcal{A}_\gamma$.

Quotient gradings

- For a Γ-grading $\mathcal{A} = \bigoplus_{\gamma\in\Gamma} \mathcal{A}_\gamma$ and $N \lhd \Gamma$ there is a natural quotient Γ/N-grading given by

$$\mathcal{A} = \bigoplus_{\bar{\gamma}\in\Gamma/N} \mathcal{A}_{\bar{\gamma}},$$

where $\mathcal{A}_{\bar{\gamma}} := \bigoplus_{\gamma\in\bar{\gamma}} \mathcal{A}_\gamma$.

- A main motivation here is that quotient gradings admit a key role in the study of the intrinsic fundamental group (Cibils, Redondo and Solotar) of an algebra $\mathcal{A}$ which is essentially the inverse limit of a diagram whose objects are groups which grade $\mathcal{A}$ in a connected way, and whose morphisms are group epimorphisms which correspond to quotient gradings between these gradings.

Example of a quotient grading

$\mathcal{A} = \bigoplus_{\bar{\gamma}\in\Gamma/N} \mathcal{A}_{\bar{\gamma}}$, where $\mathcal{A}_{\bar{\gamma}} := \bigoplus_{\gamma\in\bar{\gamma}} \mathcal{A}_{\gamma}$.

Example of a quotient grading

$\mathcal{A} = \bigoplus_{\bar{\gamma}\in\Gamma/N} \mathcal{A}_{\bar{\gamma}}$, where $\mathcal{A}_{\bar{\gamma}} := \bigoplus_{\gamma\in\bar{\gamma}} \mathcal{A}_{\gamma}$.

- Consider the $G = C_2 \times C_2 = \langle\sigma\rangle \times \langle\tau\rangle$ twisted grading of $\mathcal{A} = M_2(\mathbb{C})$. determined by

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, u_\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, u_{\sigma\tau} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Example of a quotient grading

$\mathcal{A} = \bigoplus_{\bar{\gamma} \in \Gamma/N} \mathcal{A}_{\bar{\gamma}}$, where $\mathcal{A}_{\bar{\gamma}} := \bigoplus_{\gamma \in \bar{\gamma}} \mathcal{A}_{\gamma}$.

- Consider the $G = C_2 \times C_2 = \langle \sigma \rangle \times \langle \tau \rangle$ twisted grading of $\mathcal{A} = M_2(\mathbb{C})$. determined by

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, u_\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, u_{\sigma\tau} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- For $N = \langle \tau \rangle$ we get a G/N-grading determined by

$$\mathcal{A}_{\bar{e}} = \text{span}\{E_{11}, E_{22}\}, \quad \mathcal{A}_{\bar{\sigma}} = \text{span}\{E_{12}, E_{21}\}.$$

Example of a quotient grading

$\mathcal{A} = \bigoplus_{\bar{\gamma}\in\Gamma/N} \mathcal{A}_{\bar{\gamma}}$, where $\mathcal{A}_{\bar{\gamma}} := \bigoplus_{\gamma\in\bar{\gamma}} \mathcal{A}_{\gamma}$.

- Consider the $G = C_2 \times C_2 = \langle\sigma\rangle \times \langle\tau\rangle$ twisted grading of $\mathcal{A} = M_2(\mathbb{C})$. determined by

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, u_\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, u_{\sigma\tau} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- For $N = \langle\tau\rangle$ we get a G/N-grading determined by

$$\mathcal{A}_{\bar{e}} = \text{span}\{E_{11}, E_{22}\}, \quad \mathcal{A}_{\bar{\sigma}} = \text{span}\{E_{12}, E_{21}\}.$$

- We conclude that this quotient of a twisted grading is an elementary crossed product C_2-grading determined by $(\bar{e}, \bar{\sigma})$.

Question

For which groups H of order n, the associated elementary crossed product H-grading of $M_n(\mathbb{C})$ is a quotient grading of a twisted grading $\mathbb{C}^\alpha G$ of $M_n(\mathbb{C})$ for some group of central type G of order n^2 and a nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$?

Question

For which groups H of order n, the associated elementary crossed product H-grading of $M_n(\mathbb{C})$ is a quotient grading of a twisted grading $\mathbb{C}^\alpha G$ of $M_n(\mathbb{C})$ for some group of central type G of order n^2 and a nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$?

Remark

Clearly, just by order considerations, in those cases we have $|N| = |H \cong G/N| = \sqrt{|G|}$.

Question

For which groups H of order n, the associated elementary crossed product H-grading of $M_n(\mathbb{C})$ is a quotient grading of a twisted grading $\mathbb{C}^\alpha G$ of $M_n(\mathbb{C})$ for some group of central type G of order n^2 and a nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$?

Remark

Clearly, just by order considerations, in those cases we have $|N| = |H \cong G/N| = \sqrt{|G|}$.

Liebler, Yellen (1979) and Howlett, Isaacs (1982)

Any group of central type is solvable.

Question

For which groups H of order n, the associated elementary crossed product H-grading of $M_n(\mathbb{C})$ is a quotient grading of a twisted grading $\mathbb{C}^{\alpha}G$ of $M_n(\mathbb{C})$ for some group of central type G of order n^2 and a nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$?

Remark

Clearly, just by order considerations, in those cases we have $|N| = |H \cong G/N| = \sqrt{|G|}$.

Liebler, Yellen (1979) and Howlett, Isaacs (1982)

Any group of central type is solvable.

Hence, in the above question a necessary condition for H is to be solvable.

Bahturin, Zaicev (2002) and Năstăsescu, van Oystaeyen (2004)

Any Γ-grading of $M_n(\mathbb{C})$ is graded isomorphic to a graded tensor product $M_t(\mathbb{C}) \otimes \mathbb{C}^{\alpha}G$ where $M_t(\mathbb{C})$ is equipped with an elementary grading, G a subgroup of central type of Γ and α is nondegenerate.

Bahturin, Zaicev (2002) and Năstăsescu, van Oystaeyen (2004)

Any Γ-grading of $M_n(\mathbb{C})$ is graded isomorphic to a graded tensor product $M_t(\mathbb{C}) \otimes \mathbb{C}^{\alpha}G$ where $M_t(\mathbb{C})$ is equipped with an elementary grading, G a subgroup of central type of Γ and α is nondegenerate.

- By the above, in order to understand quotient gradings of $M_n(\mathbb{C})$ it is sufficient to understand quotient gradings of twisted gradings.

Bahturin, Zaicev (2002) and Năstăsescu, van Oystaeyen (2004)

Any Γ-grading of $M_n(\mathbb{C})$ is graded isomorphic to a graded tensor product $M_t(\mathbb{C}) \otimes \mathbb{C}^\alpha G$ where $M_t(\mathbb{C})$ is equipped with an elementary grading, G a subgroup of central type of Γ and α is nondegenerate.

- By the above, in order to understand quotient gradings of $M_n(\mathbb{C})$ it is sufficient to understand quotient gradings of twisted gradings.
- This is because quotient H/N-grading of elementary H gradings which correspond to a tuple $(h_1, h_2, \ldots, h_n) \in H^n$ is given simply by taking the appropriate elements in the tuple $(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n) \in (H/N)^n$.

Quotients of twisted gradings

- Let $\mathbb{C}^\alpha G = \mathcal{A} = M_n(\mathbb{C})$ and let $N \lhd G$. Then the base algebra of the quotient G/N grading is $\mathcal{A}_{\bar{e}} = \mathbb{C}^\alpha N$.

Quotients of twisted gradings

- Let $\mathbb{C}^\alpha G = \mathcal{A} = M_n(\mathbb{C})$ and let $N \lhd G$. Then the base algebra of the quotient G/N grading is $\mathcal{A}_{\bar{e}} = \mathbb{C}^\alpha N$.
- The twisted group algebra $\mathbb{C}^\alpha G$ determines a G/N-action on the set $\text{Irr}(\mathbb{C}^\alpha N)$ of isomorphism types of irreducible $\mathbb{C}^\alpha N$-modules (alternatively, the set $\text{Irr}(N, \alpha)$ of irreducible α-projective representations of N).

Quotients of twisted gradings

- Let $\mathbb{C}^\alpha G = \mathcal{A} = M_n(\mathbb{C})$ and let $N \lhd G$. Then the base algebra of the quotient G/N grading is $\mathcal{A}_{\bar{e}} = \mathbb{C}^\alpha N$.
- The twisted group algebra $\mathbb{C}^\alpha G$ determines a G/N-action on the set $\text{Irr}(\mathbb{C}^\alpha N)$ of isomorphism types of irreducible $\mathbb{C}^\alpha N$-modules (alternatively, the set $\text{Irr}(N, \alpha)$ of irreducible α-projective representations of N).
- For $\alpha \in Z^2(G, \mathbb{C}^*)$ nondegenerate, this action is transitive, and consequently all the irreducible $\mathbb{C}^\alpha N$-modules are of the same dimension.

Quotients of twisted gradings

- Let $\mathbb{C}^\alpha G = \mathcal{A} = M_n(\mathbb{C})$ and let $N \lhd G$. Then the base algebra of the quotient G/N grading is $\mathcal{A}_{\bar{e}} = \mathbb{C}^\alpha N$.
- The twisted group algebra $\mathbb{C}^\alpha G$ determines a G/N-action on the set $\text{Irr}(\mathbb{C}^\alpha N)$ of isomorphism types of irreducible $\mathbb{C}^\alpha N$-modules (alternatively, the set $\text{Irr}(N, \alpha)$ of irreducible α-projective representations of N).
- For $\alpha \in Z^2(G, \mathbb{C}^*)$ nondegenerate, this action is transitive, and consequently all the irreducible $\mathbb{C}^\alpha N$-modules are of the same dimension.
- For $[M] \in \text{Irr}(\mathbb{C}^\alpha N)$ let $\mathcal{I}_M = \mathcal{I}_{\mathbb{C}^\alpha G}(M) < G/N$ be its stabilizer subgroup (or the *inertia* subgroup) under the G/N-action.

Theorem

Let $\mathbb{C}^\alpha G \cong M_n(\mathbb{C})$, let $N \lhd G$ and let $[M] \in Irr(\mathbb{C}^\alpha N)$. The G/N quotient grading admits a decomposition as

$$M_t(\mathbb{C}) \otimes (\mathbb{C}^\omega \mathcal{I}_M),$$

where $M_t(\mathbb{C})$ possess an elementary grading and $[\omega] \in H^2(\mathcal{I}_M, \mathbb{C}^*)$ is Mackey's obstruction cohomology class which corresponds to $[M]$.

Theorem

Let $\mathbb{C}^\alpha G \cong M_n(\mathbb{C})$, let $N \lhd G$ and let $[M] \in Irr(\mathbb{C}^\alpha N)$. The G/N quotient grading admits a decomposition as

$$M_t(\mathbb{C}) \otimes (\mathbb{C}^\omega \mathcal{I}_M),$$

where $M_t(\mathbb{C})$ possess an elementary grading and $[\omega] \in H^2(\mathcal{I}_M, \mathbb{C}^*)$ is Mackey's obstruction cohomology class which corresponds to $[M]$.

Corollary

With the above notation the quotient grading is elementary if and only if $\mathcal{I}_M$ is trivial, that is the action of G/N on $Irr(\mathbb{C}^\alpha N)$ is free.

- As stated above, the quotient G/N-grading is elementary if and only if G/N acts freely on $\text{Irr}(\mathbb{C}^{\alpha}N)$. In this case, since the action is also transitive, we have $|\text{Irr}(\mathbb{C}^{\alpha}N)| = |G/N|$.

- As stated above, the quotient G/N-grading is elementary if and only if G/N acts freely on $\mathrm{Irr}(\mathbb{C}^{\alpha}N)$. In this case, since the action is also transitive, we have $|\mathrm{Irr}(\mathbb{C}^{\alpha}N)| = |G/N|$.
- Consequently, for $|N| = \sqrt{|G|}$, the G/N-grading is elementary (and hence elementary crossed product) if and only if $|\mathrm{Irr}(\mathbb{C}^{\alpha}N)| = |N|$ which happens if and only if N is abelian and the restriction of α to N is trivial (that is N is isotropic with respect to α).

Corollary

Let $\alpha \in Z^2(G, \mathbb{C}^*)$ *be nondegenerate and let* $N \lhd G$. *The* G/N*-quotient grading of* $\mathbb{C}^\alpha G$ *is an elementary crossed product grading if and only if* N *is isotropic with respect to* α *and* $|N| = \sqrt{|G|}$.

Corollary

Let $\alpha \in Z^2(G, \mathbb{C}^*)$ be nondegenerate and let $N \lhd G$. The G/N-quotient grading of $\mathbb{C}^\alpha G$ is an elementary crossed product grading if and only if N is isotropic with respect to α and $|N| = \sqrt{|G|}$.

Example

Let $G = C_n \times C_n = \langle\sigma\rangle \times \langle\tau\rangle$, let η_n be an n-th primitive root of unity and consider the G-twisted grading of $\mathcal{A} = M_n(\mathbb{C})$ which corresponds to $[\alpha] \in H^2(G, \mathbb{C}^*)$ defined by

$$u_\tau^i = u_{\tau^i}, \quad u_\sigma^i = u_{\sigma^i}, \quad u_\tau u_\sigma = \eta_n u_\sigma u_\tau.$$

Now, let $N = \langle\tau\rangle \cong C_n$. Then, the G/N quotient grading is an elementary crossed product grading.

Theorem

A subgroup L of G which is isotropic with respect to nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$ such that $|L| = \sqrt{|G|}$ is called Lagrangian with respect to α.

Theorem

A subgroup L of G which is isotropic with respect to nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$ such that $|L| = \sqrt{|G|}$ is called Lagrangian with respect to α.

Ben david, Ginosar (2009)

A group H is IYB (that is a multiplicative group of a brace) if and only if there exists a group of central type G and a normal Lagrangian $L \lhd G$ with respect to a nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$ such that $H \cong G/L$.

Theorem

A subgroup L of G which is isotropic with respect to nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$ such that $|L| = \sqrt{|G|}$ is called Lagrangian with respect to α.

Ben david, Ginosar (2009)

A group H is IYB (that is a multiplicative group of a brace) if and only if there exists a group of central type G and a normal Lagrangian $L \lhd G$ with respect to a nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$ such that $H \cong G/L$.

So we have the following characterization of IYB groups.

Theorem

A subgroup L of G which is isotropic with respect to nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$ such that $|L| = \sqrt{|G|}$ is called Lagrangian with respect to α.

Ben david, Ginosar (2009)

A group H is IYB (that is a multiplicative group of a brace) if and only if there exists a group of central type G and a normal Lagrangian $L \lhd G$ with respect to a nondegenerate $\alpha \in Z^2(G, \mathbb{C}^*)$ such that $H \cong G/L$.

So we have the following characterization of IYB groups.

Theorem

A group H of order n is IYB if and only if its corresponding elementary crossed-product grading is a quotient of some twisted grading of $M_n(\mathbb{C})$.

Thank you.