Indecomposable solutions of the Yang–Baxter equation and generators of skew braces

Joint work with Marco Castelli

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June 19, 2023

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Preliminaries and notation

Definition

A skew (left) brace is a triple $(A, +, \circ)$ with A a set, $(A, +)$ and $(A, \circ)$ group structures, and for all $a, b, c \in A$,

$$a \circ (b + c) = a \circ b - a + a \circ c.$$

If $(A, +)$ is abelian, then we say that A is a brace

Preliminaries and notation

We define $\lambda_a(b) = -a + a \circ b$ and in this way obtain an action

$$\lambda : (A, \circ) \to \mathrm{Aut}(A, +) : a \mapsto \lambda_a.$$

Strong left ideals and ideals

Definition

Let A be a skew brace, a subgroup I of $(A, +)$ is

1. A strong left ideal if $\lambda_a(I) \subseteq I$ for all $a \in A$ and I is normal in $(A, +)$.
2. An ideal if $\lambda_a(I) \subseteq I$ for all $a \in A$ and I is normal in $(A, +)$ and $(A, \circ)$.

Multipermutation skew braces

For any skew brace A, its socle

$$\operatorname{Soc}(A) = \ker \lambda \cap Z(A, +),$$

is an ideal of A. Inductively we define

$$\operatorname{Ret}^0(A) = A, \quad \operatorname{Ret}^{i+1}(A) = \operatorname{Ret}^i(A)/\operatorname{Soc}(\operatorname{Ret}^i(A)).$$

If there exists some n such that $\operatorname{Ret}^n(A) = 0$, then we say that A is multipermutation.

Preliminaries and notation

Definition

A set-theoretical solution to the YBE is a pair (X, r) with X a non-empty set and $r : X \times X \to X \times X$ a bijective map such that

$$r_1 r_2 r_1 = r_2 r_1 r_2,$$

with $r_1 = r \times \mathrm{id}_X$ and $r_2 = \mathrm{id}_X \times r$. A solution (X, r) is

- non-degenerate if all σ_x, τ_x are invertible maps, with $r(x, y) = (\sigma_x(y), \tau_y(x))$,

By a solution we mean a non-degenerate set-theoretical solution to the YBE.

Preliminaries and notation

The orbits of a solution (X, r) are the smallest partition of X which is invariant under σ_x and τ_x for all $x \in X$. A solution is indecomposable if it contains a unique orbit.
Equivalently, a solution is indecomposable if there exists no non-trivial partition $X = X_1 \cup X_2$ such that $r(X_i \times X_j) = X_j \times X_i$ for all $i, j \in \{1, 2\}$.

Preliminaries and notation

Let (X, r) be a solution. Define

$$(G(X,r),\circ) = \langle x \in X \mid x \circ y = \sigma_x(y) \circ \tau_y(x), \forall x, y \in X \rangle$$

then there is canonical additive structure $(G(X,r),+,\circ)$ is a skew brace.

Preliminaries and notation

Also define

$$(\mathcal{G}(X,r),\circ) = \langle (\sigma_x, \tau_x^{-1}) \mid x \in X\rangle \subseteq \mathrm{Perm}(X) \times \mathrm{Perm}(X)\rangle$$

then there exists a canonical additive structure such that $(\mathcal{G}(X,r),+,\circ)$ is a skew brace such $x \mapsto (\sigma_x, \tau_x^{-1})$ extends to a skew brace homomorphism

$$(G(X,r),+,\circ) \to (\mathcal{G}(X,r),+,\circ).$$

Preliminaries and notation

Proposition ([GV17])

If A is a skew brace then $r : A^2 \to A^2$ with

$$r_A(a, b) = (\lambda_a(b), \overline{\lambda_a(b)} \circ a \circ b),$$

yields a solution (A, r_A).

Preliminaries and notation

Theorem ([BCJ16, Bac18])

Given a skew brace A, then it is possible to construct all solutions (X, r) such that $\mathcal{G}(X, r) \cong A$.

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Given a skew brace A, then it is possible to construct all solutions (X, r) such that $\mathcal{G}(X, r) \cong A$.

Central question: what properties of (X, r) correspond to properties of $\mathcal{G}(X, r)$ and vice versa.

Cycle bases

Definition

We define an action $\theta : (A, +) \rtimes (A, \circ) \to \mathrm{Aut}(A, +)$ where for all $a, b, c \in A$,

$$\theta_{(a,b)}(c) = a + \lambda_b(c) - a.$$

The orbits of θ are called the orbits of A.

Definition

A subset X of a skew brace A is cycle base if $\theta_{(a,b)}(X) \subseteq X$ for all $a, b \in A$ and $\langle X \rangle_+ = A$. If X consists of a single orbit then it is a transitive cycle base

Cycle bases for skew braces have not explicitly been defined before, but have appeared in literature.

Why cycle bases?

A cycle base X of A gives rise to a solution in two different ways:

1. The solution (A, r_A) restricts to X. The orbits of $(X, r_A|_{X\times X})$ coincide with the orbits of A contained in X.
2. The construction of Bachiller, Cedó and Jespers starts with a cycle base.

Why transitive cycle bases?

A transitive cycle base X of A gives rise to indecomposable solutions:

1. The restriction of (A, r_A) to X is indecomposable.
2. To obtain an indecomposable solution through the construction by Bachiller, Cedó and Jespers, we first of all need a transitive cycle base.

When does a skew brace admit a transitive cycle base?

Theorem ([SS18, Rum20])

Let A be a multipermutation brace. Then A admits a transitive cycle base if and only if it is one-generated. Every element contained in a transitive cycle base generates A and every orbit containing a generator is a transitive cycle base.

Is there a general connection between cycle bases and generators?

Definition

Let A be a skew brace, a subgroup I of $(A, +)$ is

1. A strong left ideal if $\lambda_a(I) \subseteq I$ for all $a \in A$ and I is normal in $(A, +)$.
2. An ideal if $\lambda_a(I) \subseteq I$ for all $a \in A$ and I is normal in $(A, +)$ and $(A, \circ)$.

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Proposition ([CT])

Let X be a union of orbits of a skew brace A and let Y be a set of representatives of X. Then X is a cycle base if and only if Y generates A as a strong left ideal.

Is there a general connection between cycle bases and generators?

Corollary ([CT])

The minimal number of orbits in a cycle base of A coincides with the minimal number of generators of A as a strong left ideal. In particular, a skew brace admits a transitive cycle base if and only if it is one-generated as a strong left ideal.

When does a skew brace admit a transitive cycle base?

Theorem ([SS18, Rum20])

Let A be a multipermutation brace. Then A admits a transitive cycle base if and only if it is one-generated. Every element contained in a transitive cycle base generates A and every orbit containing a generator is a transitive cycle base.

When does a skew brace admit a transitive cycle base?

Theorem ([SS18, Rum20])

Let A be a multipermutation brace. A is one-generated as a strong left ideal if and only if it is one-generated as a brace. Every element generating A as a strong left ideal also generates A as a brace.

When does a skew brace admit a transitive cycle base?

Theorem ([SS18, Rum20, CT])

Let A be a multipermutation skew brace and $X \subseteq A$. Then X generates A as a strong left ideal if and only if X generates A as a skew brace.

Does this only work for multipermutation skew braces?

Similar related results can be obtained for left nilpotent skew braces and centrally nilpotent skew braces, relating generating sets of skew braces.

Thank you

Thank you for listening!

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