Derived-Indecomposable Solutions, Skew Braces and Presentations

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Groups, Rings and the Yang-Baxter Equation **2023**

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On derived-indecomposable solutions of the Yang-Baxter equation

arXiv: 2210.08598

M.T.

The structure skew brace associated with a finite non-degenerate solution of the Yang-Baxter equation is finitely presented

Proc. Amer. Math. Soc.

to appear

Let B be a set. If $(B, +)$ and $(B, \circ)$ are (not necessarily abelian) groups, the triple $(B, +, \circ)$ is a **skew (left) brace** if

$$a \circ (b + c) = a \circ b - a + a \circ c$$

for all $a, b, c \in B$.

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for all $a, b, c \in B$.

$0 = 1$ is the identity of the **additive group** $(B, +)$ and the identity of the **multiplicative group** $(B, \circ)$

$-a$ and a^{-1} are the **inverses** of a in $(B, +)$ and $(B, \circ)$, respectively

$na = a + a + \ldots + a$ and $a^n = a \circ a \circ \ldots \circ a$

Let B be a set. If $(B,+)$ and $(B,\circ)$ are (not necessarily abelian) groups, the triple $(B,+,\circ)$ is a **skew (left) brace** if

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for all $a,b,c\in B$.

$\boldsymbol{\lambda} : \mathbf{a} \in (B,\circ) \mapsto \left(\boldsymbol{\lambda}_{\mathbf{a}} : b\mapsto \lambda_a(b)=-a+a\circ b\right)\in \mathrm{Aut}(B,+)$ is a group homomorphism

- $a+b=a\circ\lambda_a^{-1}(b)$
- $a\circ b=a+\lambda_a(b)$
- $-a=\lambda_a\left(a^{-1}\right)$

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$a * b = \lambda_a(b) - b = -a + a \circ b - b$

- $a * (b + c) = a * b + b + a * c - b$
- $(a \circ b) * c = a * (b * c) + b * c + a * c$

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$G = (B,+) \rtimes (B,\circ)$ with $(a,b)(c,d) = \left(a + \lambda_b(c), b \circ d\right)$

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$$\left[(0, a), (b, 0)\right] = (a * b, 0)$$

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An **ideal** of B is subset X of B which is a normal subgroup of both $(B, +)$ and $(B, \circ)$, and $\lambda_b(X) \subseteq X$.

A skew brace B is **finitely generated** if there is a set $S = \{x_1, \ldots, x_n\}$ such that B is the smallest sub-skew brace containing S with respect to the inclusion.

Presentations

Juan Orza

> *A construction of the free skew-brace*
>
> **ArXiv**: 2002.12131

Juan Orza

A construction of the free skew-brace

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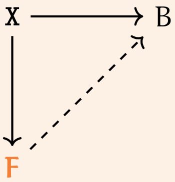

- F is a free skew brace on the set X
- B is any skew brace

Let **B** be a skew brace. A **presentation** of **B** is an exact sequence of skew braces

$$0 \to R \to F \xrightarrow{\theta} \mathbf{B} \to 0, \qquad (\star)$$

where F is a free brace over some set X.

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where F is a free brace over some set $\mathtt{X}$.

Assume **B** is finitely generated.

Then $(\star)$ is a **finite presentation** of **B** if:

- $\mathtt{X} = \{\mathtt{x}_1, \ldots, \mathtt{x}_m\}$
- $\mathrm{Ker}(\theta)$ is finitely generated as an ideal of F (by the elements $\rho_1, \ldots, \rho_n$)

B is **finitely presented** by $\theta(\mathtt{x}_1), \ldots, \theta(\mathtt{x}_m)$ subject to the relations $\rho_1 = \ldots = \rho_n = 1$.

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B is **finitely presented** by $\theta(x_1), \dots, \theta(x_m)$ subject to the relations $\rho_1 = \dots = \rho_n = 1$.

The **definition** is independent of the presentation

Examples

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- Trivial skew braces whose underlying group is finitely presented as a group

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(MT 2023)
Let $\mathbf{B}$ be a **skew brace** and $\mathbf{I}$ an ideal of $\mathbf{B}$. If

$(\mathbf{I}, +)$, $(\mathbf{I}, \circ)$, $(\mathbf{B}/\mathbf{I}, \circ)$ are **finitely generated**, and

$\mathbf{I}$, $\mathbf{B}/\mathbf{I}$ and $(\mathbf{B}/\mathbf{I}, +)$ are **finitely presented**,

then $\mathbf{B}$ is **finitely presented**.

(**MT** 2023)
The **structure skew brace**

associated with a finite non-degenerate solution of the **YBE**

is **finitely presented**

If $(\mathbf{X}, \mathbf{r})$ is a **non-degenerate solution** of the **YBE**, then there is a unique skew brace structure over the structure group

$$\mathbf{G(X,r)} = \langle X \,|\, xy = \sigma_x(y)\tau_y(x),\ x, y \in X \rangle$$

such that

$$r_{G(X,r)}(\iota \times \iota) = (\iota \times \iota)r,$$

where $\iota : X \longrightarrow G(X,r)$ is the canonical map: the multiplicative group of this skew brace is $G(X,r)$ and the additive is

$$\langle X \,|\, x + \sigma_x(y) = \sigma_x(y) + \sigma_{\sigma_x(y)}\big(\tau_y(x)\big)\ \forall\, x, y \in X \rangle.$$

This is the **structure skew brace** of $(\mathbf{X}, \mathbf{r})$.

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The **socle** $\mathrm{Soc}(\mathbf{B})$ of a skew brace $\mathbf{B}$ is the intersection of $\mathrm{Ker}(\boldsymbol{\lambda})$ and $Z(\mathbf{B}, +)$.

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Let $\mathbf{B}$ be a **finitely generated** skew brace such that

$\mathbf{B}/\mathrm{Soc}(\mathbf{B})$ is **finite**, and

$(\mathbf{B}, \circ)$ is **virtually abelian**,

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- V. Lebed – L. Vendramin: "On structure groups of set-theoretic solutions to the Yang-Baxter equation", *Proc. Edinb. Math. Soc.* (2) 62 (2019), 683–717.

(MT 2023)
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then **B** is **finitely presented**.

$\mathrm{Soc}(\mathbf{B})$ contains an ideal **I** such that:

- **I** is finitely generated;
- $\mathbf{B}/\mathbf{I}$ is finite;
- every element of **I** has finitely many **conjugates** in **B**.

Let **B** be a **skew brace**. If $\mathbf{b} \in \mathbf{B}$, then element of the form

$$\mathbf{b} * c,$$

$$c * \mathbf{b},$$

$$c + \mathbf{b} - c,$$

$$c^{-1} \circ \mathbf{b} \circ c$$

($c \in \mathbf{B}$) will be referred to as a **conjugate** of b in B.

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I. Colazzo – M. Ferrara – M.T.: “On derived-indecomposable solutions of the Yang-Baxter equation”, to appear

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(**MT** 2023)

Let **B** be a **skew brace** and let **I** be an ideal of **B**. If

- **B** is **finitely generated** and
- **B**/**I** is **finitely presented**,

then **I** is **finitely generated as an ideal** of **B**.

Let **B** be a skew brace. We recursively define the **upper annihilator series** of **B** as follows. Put $\operatorname{Ann}_0(\mathbf{B}) = \{0\}$ and

$$\operatorname{Ann}_1(\mathbf{B}) = \operatorname{Ann}(\mathbf{B}) = \operatorname{Soc}(\mathbf{B}) \cap Z(\mathbf{B}, \circ).$$

If α is an ordinal number, put

$$\operatorname{Ann}_{\alpha+1}(\mathbf{B}) / \operatorname{Ann}_{\alpha}(\mathbf{B}) = \operatorname{Ann}\left(\mathbf{B} / \operatorname{Ann}_{\alpha}(\mathbf{B})\right).$$

If μ is a limit ordinal, put

$$\operatorname{Ann}_{\mu}(\mathbf{B}) = \bigcup_{\gamma < \mu} \operatorname{Ann}_{\gamma}(\mathbf{B}).$$

The last term of the upper socle series is the **hyper-annihilator** of **B** and is denoted by $\overline{\operatorname{Ann}}(\mathbf{B})$.

Let $\mathbf{B}$ be a skew brace. If $\mathbf{B} = \overline{\mathrm{Ann}}(\mathbf{B})$ we say that $\mathbf{B}$ is **annihilator hypercentral**.

Let $\mathbf{B}$ be a skew brace. If $\mathbf{B} = \mathrm{Ann}_n(\mathbf{B})$ we say that $\mathbf{B}$ is **annihilator nilpotent**.

Let **B** be a skew brace. If $\mathbf{B} = \overline{\mathrm{Ann}}(\mathbf{B})$ we say that **B** is **annihilator hypercentral**.

Let **B** be a skew brace. If $\mathbf{B} = \mathrm{Ann}_n(\mathbf{B})$ we say that **B** is **annihilator nilpotent**.

- E. Jespers – A. Van Antwerpen – L. Vendramin: "Nilpotency of skew braces and multipermutation solutions of the Yang-Baxter equation"; *arXiv*:2205.01572

Let **B** be an **annihilator nilpotent** skew brace satisfying ACC. The following statements are equivalent:

- **B** is **finitely generated**;
- $(\mathbf{B}, +)$ is **finitely generated**;
- $(\mathbf{B}, \circ)$ is **finitely generated**.

(MT 2023)
Let $\mathbf{B}$ be an **annihilator hypercentral** skew brace.
The following statements are equivalent:

- $\mathbf{B}$ is **finitely generated**;
- $\mathbf{B}$ is **finitely presented**;
- $(\mathbf{B}, +)$ is **finitely generated**;
- $(\mathbf{B}, \circ)$ is **finitely generated**.

Let $\mathbf{B}$ be a skew brace. Describe the connection between the following statements:

- $\mathbf{B}$ is **finitely presented**;
- $(\mathbf{B}, +)$ is **finitely presented**;
- $(\mathbf{B}, \circ)$ is **finitely presented**.

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Do we really need all the hypotheses in results such as this one?

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. . . *Fin*

. . . *Thank You* . . .