Irreducible representations of the Hecke–Kiselman algebras

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Definition (Ganyushkin, Mazorchuk)

Let Θ be a simple oriented graph with n vertices. Then the corresponding Hecke–Kiselman monoid HK_Θ is the monoid generated by idempotents $x_1, \ldots, x_n$ such that:

1) if the vertices i, j are not connected in Θ, then $x_i x_j = x_j x_i$,
2) if i, j are connected by an arrow $i \to j$ in Θ, then $x_i x_j x_i = x_j x_i x_j = x_i x_j$.

If K is a field then $K[\mathrm{HK}_\Theta]$ denotes the corresponding monoid algebra, called the Hecke–Kiselman algebra.

$\rightsquigarrow$ natural quotient of the 0-Hecke monoids

Theorem (Ganyushkin, Mazorchuk)

1) Monoid HK_Θ is finite $\iff$ the graph Θ is acyclic.
2) Finite Hecke–Kiselman monoids are $\mathcal{J}$-trivial, that is $\mathrm{HK}_\Theta \, w \, \mathrm{HK}_\Theta = \mathrm{HK}_\Theta \, v \, \mathrm{HK}_\Theta$ implies that $w = v$ in HK_Θ.

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Algebra $K[C_n]$ associated to an oriented cycle

Monoid C_n for any $n \geqslant 3$ is given by the presentation

$$\langle x_1, \ldots, x_n : x_i^2 = x_i,\ x_i x_{i+1} = x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ for } i = 1, \ldots, n, \\ x_i x_j = x_j x_i \text{ for } n-1 > i-j > 1\rangle$$

What is known about $K[C_n]$?

- (Denton) C_n is a $\mathcal{J}$-trivial monoid.
- (Męcel, Okniński) $K[C_n]$ is a PI-algebra of Gelfand–Kirillov dimension one.
- (Okniński, W.) Algebra $K[C_n]$ is Noetherian and semiprime.

Useful tool: semigroups of matrix type

Definition

If S is a semigroup, A, B are nonempty sets and $P = (p_{ba})$ is a $B \times A$ -matrix with entries in S^0, then the semigroup of matrix type $\mathcal{M}^0(S, A, B; P)$ over S is the set of all matrices of size $A \times B$ with at most one nonzero entry with the operation

$$M \cdot N = M \circ P \circ N$$

for every matrices M and N, where $\circ$ is standard matrix multiplication.

Ideal chain and matrix structures inside C_n

Theorem

C_n has a chain of ideals

$$\emptyset = I_{n-2} \lhd I_{n-3} \lhd \cdots \lhd I_0 \lhd I_{-1} \lhd C_n,$$

with the following properties

1) for $i = 0, \ldots, n-2$ there exist semigroups of matrix type $M_i = \mathcal{M}^0(S_i, A_i, B_i; P_i)$, such that $M_i \subset I_{i-1}/I_i$ (we agree that $I_{n-3}/\emptyset = I_{n-3} \cup \{\theta\}$), where S_i is the infinite cyclic semigroup, P_i is a square symmetric matrix of size $B_i \times A_i$ and with coefficients in $S_i^1 \cup \{\theta\}$;
2) $|A_i| = |B_i| = \binom{n}{i+1}$ for every $i = 0, \ldots, n-2$;
3) for $i = 1, \ldots, n-2$ the sets $(I_{i-1}/I_i) \setminus M_i$ are finite and C_n/I_{-1} is a finite semigroup.

Motivation: irreducible representations of finite monoids

Every finite monoid M admits a chain of principal ideals

$$\emptyset = M_k \lhd M_{k-1} \lhd \cdots \lhd M_1 = M$$

such that each factor is either null semigroup or 0-simple semigroup, which is isomorphic to $\mathcal{M}^0(G, X, Y; P)$, where G is a group.

Clifford–Munn–Ponizovskii theorem

1) $\left\{\text{irreducible representations}\right\} \leftrightsquigarrow \left\{\begin{array}{c}\text{irreducible representations of}\\ \text{0-simple factors}\end{array}\right\}$

2) $\left\{\begin{array}{c}\text{irreducible representations}\\ \text{of 0-simple semigroup}\\ \mathcal{M}^0(G, X, Y; P)\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{c}\text{irreducible representations of}\\ \text{the maximal subgroup G}\end{array}\right\}$

Case of finite $\mathcal{J}$-trivial monoids

$$\left\{\begin{array}{c}\text{irreducible representations}\\ \text{of M}\end{array}\right\} \leftrightsquigarrow \left\{\text{idempotents of M}\right\}$$

Irreducible representations of the algebra $K[C_n]$

Theorem

Let $\varphi : K[C_n] \longrightarrow M_j(K)$ be an irreducible representation of the Hecke–Kiselman algebra $K[C_n]$ over an algebraically closed field K. If $\varphi(K[I_{n-3}]) \neq 0$ set $i = n - 2$. Otherwise take the minimal $i \in \{-1, \ldots, n-3\}$ such that $\varphi(K[I_i]) = 0$.

1) If $i \geqslant 0$ and $\varphi(K[M_i]) \neq 0$, then the representation φ is <u>induced by a representation of $K[M_i]$</u>.
2) If ($i \geqslant 0$ and $\varphi(K[M_i]) = 0$) or $i = -1$, then the representation φ is one-dimensional and <u>induced by an idempotent</u> $e \in I_{i-1} \setminus I_i$ or $e \in C_n \setminus I_{-1}$, respectively.

$\rightsquigarrow$ characterization of all idempotents of the monoid C_n is known

Irreducible representations of $K[M_i]$

Recall that $M_i = \mathcal{M}^0(S_i, A_i, B_i; P_i)$, where S_i is infinite cyclic semigroup generated by s_i, P_i is a $B_i \times A_i$ matrix with coefficients in $S_i^1 \cup \{\theta\}$.

$$M_i \rightsquigarrow \text{completely 0-simple closure } cl(M_i) = \mathcal{M}^0(\mathrm{gr}(s_i), A_i, B_i; P_i).$$

Theorem

Every irreducible representation of the infinite cyclic group $\mathrm{gr}(s_i)$ induces a unique irreducible representation of M_i. It is induced by an irreducible representation of $cl(M_i)$.

Conversely, every irreducible representation of M_i comes from a representation of the group $\mathrm{gr}(s_i)$, and can be uniquely extended to an irreducible representation of $cl(M_i)$.

PI Hecke–Kiselman algebras

PI Hecke–Kiselman algebras (Męcel, Okniński)

Hecke–Kiselman algebra $K[\mathrm{HK}_\Theta]$ satisfies a polynomial identity if and only if Θ does not contain two different cycles connected by an oriented path of length $k \geqslant 0$.

The radical of PI Hecke–Kiselman algebra (Okniński, W.)

Let Θ' be the subgraph of Θ obtained by deleting all arrows $x \to y$ that are not contained in any cyclic subgraph of Θ.
The Jacobson radical $J(K[\mathrm{HK}_\Theta])$ can be described. In particular

$$K[\mathrm{HK}_\Theta]/J(K[\mathrm{HK}_\Theta]) \cong K[\mathrm{HK}_{\Theta'}] \cong K[\mathrm{HK}_{\Theta_1}] \otimes \cdots \otimes K[\mathrm{HK}_{\Theta_m}],$$

where $\Theta_1, \ldots, \Theta_m$ are connected components of Θ', and algebras $K[\mathrm{HK}_{\Theta_i}]$ are isomorphic to $\underline{K \oplus K}$ or to the algebra $\underline{K[C_j], \text{ for some } j \geqslant 3,}$ for all $i = 1, \ldots, m$.

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Irreducible representations of PI Hecke–Kiselman algebras

Theorem

Every irreducible representation of $K[\mathrm{HK}_\Theta]$ is of the form

$$K[\mathrm{HK}_\Theta] \to K[\mathrm{HK}_{\Theta_1}] \otimes \cdots \otimes K[\mathrm{HK}_{\Theta_m}] \to M_{r_1}(K) \otimes \cdots \otimes M_{r_m}(K) \xrightarrow{\simeq} M_{r_1 \cdots r_m}(K),$$

where

1) the first map is the natural epimorphism onto $K[\mathrm{HK}_\Theta]/J(K[\mathrm{HK}_\Theta])$,
2) the second homomorphism is $\psi_1 \otimes \cdots \otimes \psi_m$ for some irreducible representations $\psi_i : K[\mathrm{HK}_{\Theta_i}] \to M_{r_i}(K)$, $i = 1, \ldots, m$.

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Thank you!