The Isomorphism Problem for Rational Group Rings of Metacyclic Groups

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A joint work with **Ángel del Río Mateos**

Groups, Rings and the Yang-Baxter Equation
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The Isomorphism Problem for Rational Group Rings

$$\mathbb{Q}G \Rightarrow G$$

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$$\mathbb{Q}G \cong \mathbb{Q}H \overset{?}{\Rightarrow} G \cong H$$

The Isomorphism Problem for Group Rings

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First Proposed by Higman in 1940.

Integral Isomorphism Problem for Group Rings

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- No, **[M. Hertweck, 2001]**

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- No, **[D. García, L. Margolis, Á. del Río, 2021]**

Metacyclic Groups

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Definition (Metacyclic)

G is metacyclic when $\exists N \trianglelefteq G$ cyclic s.t. G/N is also cyclic.

Motivation I

Theorem (Olivieri, del Rio, Simon (2004))

Let G be a finite ***metabelian*** *group and let A be a maximal abelian subgroup of G containing G'. Then* ***every Wedderburn component of $\mathbb{Q}G$ is of the form $\mathbb{Q}Ge(G, L, K)$ for subgroups L and K*** *satisfying the following conditions:*

1. L *is a maximal element in the set* $\{B \leq G : A \leq B \text{ and } B' \leq K \leq B\}$.
2. L/K *is cyclic.*

Motivation II

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"More information"

$$\mathbb{Q}Ge(G,L,K) \cong M_n(A(G,L,K))$$

$$A(G,L,K) \cong (\mathbb{Q}(\zeta_m)/F, \alpha, \zeta_m^y) = \mathbb{Q}(\zeta_m)[\overline{u} \mid \zeta_m \overline{u} = \overline{u}\zeta_m^x, \overline{u}^k = \zeta_m^y].$$

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In particular, $n = [G : N_G(K)]$, $m = [L : K]$.

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Subgroups	Wedderburn Components
$(L = G, K = \langle a \rangle)$	$\mathbb{Q}$
$(G, \langle a^2 \rangle$	$\mathbb{Q}(\zeta_3)$
$(G, \langle a^3 \rangle)$	$\mathbb{Q}(\mathrm{i})$
$(G, \langle a^4 \rangle)$	$\mathbb{Q}(\zeta_3)$
$(G, K = \langle a^6 \rangle)$	$\mathbb{Q}$
$(G, 1)$	$\mathbb{Q}(\zeta_{12})$

Main tool

$$\mathbb{Q}G \cong \mathbb{Q}H, \quad A(G, L_1, K_1) = A(H, L_2, K_2)$$

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3. Prove that a component that satisfies the conditions in (1) can **only** come from a pair as in (2).

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Theorem

Let G, H be finite metacyclic groups. Then $\mathbb{Q}G \cong \mathbb{Q}H$ *if and only if* $G \cong H$.

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p-group $\rightarrow$ nilpotent $\rightarrow$ general case

The End

Thank you very much for your attention.