Growth, dynamics and geometry of noncommutative algebras

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Groups, Rings and the Yang-Baxter Equation

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Growth of algebras

A - f.g. associative algebra over a field F
V - f.d. generating subspace of $A = F + V + V^2 + \cdots$

$$\gamma_{A,V}(n) = \dim_F (F + V + \cdots + V^n)$$

is the **growth** of A. Independent of choice of V up to asymptotic equivalence ($f \sim g$ if $f \leq g(Cn)$ and $g \leq f(Dn)$ for $n \gg 1$).

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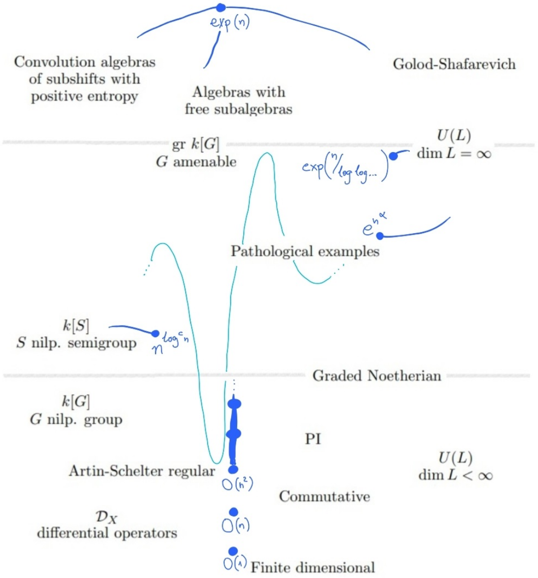
- If $\gamma_A(n)$ is **polynomially bounded**, then the Gel'fand-Kirillov dimension is:

  $$\mathrm{GKdim}(A) = \overline{\lim}_{n\to\infty} \log_n \gamma_A(n)$$

  e.g. for a commutative algebra, $\mathrm{GKdim}(A) = \mathrm{Kdim}(A)$
- If $\limsup_{n\to\infty} \frac{1}{n} \log \gamma_A(n) > 0$, A has **exponential growth** (e.g. a noncommutative free algebra)
- If γ_A is super-polynomial but subexponential, A has **intermediate growth** (e.g. $U(Vir) \sim \exp(\sqrt{n})$)

The space of growth functions



Monomial algebras and symbolic dynamics

Σ - finite alphabet, $X \subseteq \Sigma^{\mathbb{N}}$ a closed, shift-invariant subspace (subshift).
Complexity: $p_X(n) = \#\{\text{Subwords of } X \text{ of length } n\}$.

$$A_X = F\langle\Sigma\rangle / \langle\text{Monomials which do not factor any word from } X\rangle .$$

Sometimes we can localize:

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Subshift		Monomial algebra
Complexity	$\leftrightsquigarrow$	Growth
Irreducible	$\leftrightsquigarrow$	Prime
Eventually periodic	$\leftrightsquigarrow$	PI of linear growth
Minimal	$\leftrightsquigarrow$	Projectively simple

Polynomial growth: Oscillations and tensor products

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Krempa-Okniński ('87): *Each value* within $[\alpha + 2, \alpha + \beta]$ can occur as $\mathrm{GKdim}(A \otimes_F B)$.

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Question (Krempa-Okniński, '87; Krause-Lenagan, '00)

Are there such examples among semiprime algebras?

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Are there such examples among semiprime algebras?

Theorem (G.-Zelmanov, '22)

*For any $2 \leq \alpha \leq \beta$ and $\gamma \in [\alpha + 2, \alpha + \beta]$ there exist **simple** algebras $\mathrm{GKdim}(A) = \alpha, \mathrm{GKdim}(B) = \beta, \mathrm{GKdim}(A \otimes_F B) = \gamma$.*

Toeplitz subshifts with highly correlated complexity $\rightsquigarrow$ simple convolution algebras, growth controlled by the prescribed complexities

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Are there solutions to the Kurosh Problem with restricted growth?

- There exist nil algebras with GK $< \infty$ (Lenagan-Smoktunowicz, '06)
- There exist nil algebras of intermediate growth (Smoktunowicz, '14)

The **strong quantitative version** of the Kurosh Problem:

Conjecture (Zelmanov, '17; Alahmadi-Alsulami-Jain-Zelmanov, '17)

$$\left\{ \begin{matrix} \textit{Growth functions*} \\ \textit{of algebras} \end{matrix} \right\} = \left\{ \begin{matrix} \textit{Growth functions} \\ \textit{of nil algebras} \end{matrix} \right\}$$

**Except for algebras of linear growth*

The Quantitative Kurosh Problem

The growth function of *any* algebra is *increasing* $f(n) < f(n+1)$ and *submultiplicative* $f(n+m) \leq f(n)f(m)$ ('natural candidates').
For any such f there is an algebra A with $f(n) \preceq \gamma_A(n) \preceq nf(n)$.

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For any increasing, submultiplicative function f and an arbitrarily slow 'distortion' $\omega(n) \to \infty$ there exists a nil algebra/Lie algebra A such that:

$$f(n/\omega(n)) \preceq \gamma_A(n) \preceq poly(n) \cdot f(n)$$

E.g. $\exp\left(n^{\alpha+o(1)}\right)$ for any α and many other growth types.

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- We can construct monomial algebras $A = A_X$ with controlled growth (via complexity of X) Problem: Monomial algebras are not nil.
- Solution: (temporarily) give up finite generation. 'Deform' A to a locally nilpotent algebra $\widetilde{A}$
- Construct a linear map γ into $\widetilde{A}$ with fine control on its 'rate of expansiveness'. Find 'sufficiently big' nil subalgebras of $B \wr_\gamma \widetilde{A}$

The dictionary

Subshift		**Monomial algebra**
Any	$\leftrightsquigarrow$	?
Transitive	$\leftrightsquigarrow$	?

Example: $X =$ Closure of the shift orbit of $ababbabbb\ldots$

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Which ring-theoretic property 'encodes' such a structure? A graded module P is a **point module** if $H_P(t) = 1 + t + t^2 + \cdots$

$$\mathcal{P}_n(A) = \text{Moduli space of } n\text{-truncated point } A\text{-modules}$$

$$\mathcal{P}(A) = \varprojlim_n \mathcal{P}_n(A) \overset{1:1}{\longleftrightarrow} \{\text{Point } A\text{-modules}\}/\cong$$

(Pro)algebraic geometry of monomial algebras

Graded commutative algebras	$\leftrightsquigarrow$	Projective varieties
"Nice" nc algebras	$\leftrightsquigarrow$	Projective varieties + aut. twist
Monomial algebras	$\leftrightsquigarrow$	Proalgebraic varieties

Graded algebras might have no point modules at all! But monomial algebras always do (reflecting their dynamical origin).
Example: $A = F\langle x, y\rangle \rightsquigarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots$ (in fact, characterizes free alg.)

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Theorem (Bell-G., '23)

A monomial algebra is isomorphic to A_X where X is a ***transitive*** *subshift iff it admits a* ***faithful point module****.*

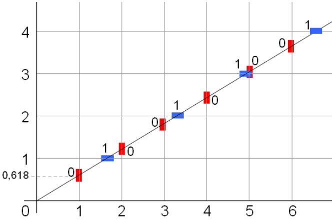
A monomial algebra is isomorphic to A_X for some subshift iff:

$$\bigcap_{\textit{Point modules}} Ann(P) = 0.$$

Monomial $\mathbb{P}^1$

Sturmian subshifts: minimal aperiodic of complexity $p_X(n) = n + 1$.
Example (Fibonacci word):

$$0100101001001\ldots$$



A_X for X Sturmian $\leftrightarrow$ Proj. simple with $H(t) = \frac{1}{(1-t)^2}$ ("Monomial $\mathbb{P}^1$")

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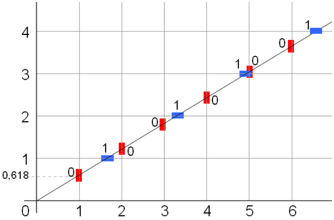

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Theorem (Bell-G., '23)

If A is a "monomial $\mathbb{P}^1$" then its proalgebraic variety of point modules is isomorphic to $\mathbb{P}^1 \cup$ Cantor set, intersecting at two points.

Thank you!

Questions?