Strong semilattices of skew braces

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Joint work with Francesco Catino and Paola Stefanelli

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Solutions of the Yang-Baxter equation

If S is a set, a map $r : S \times S \longrightarrow S \times S$ satisfying the braid relation

$$(r \times \mathrm{id}_S)\,(\mathrm{id}_S \times r)\,(r \times \mathrm{id}_S) = (\mathrm{id}_S \times r)\,(r \times \mathrm{id}_S)\,(\mathrm{id}_S \times r)$$

is called *set-theoretic solution*, or briefly *solution*, of the Yang-Baxter equation.

For a solution r, we introduce two maps $\ _a, \ _b : S \to S$ and write

$$r(a, b) = (\ _a(b), \ _b(a)),$$

for all $a, b \in S$. In particular, the solution r is said to be

- *left non-degenerate* if $\ _a$ is bijective, for every $a \in S$;
- *right non-degenerate* if $\ _b$ is bijective, for every $b \in S$;
- *non-degenerate* if r is both left and right non-degenerate.

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Solutions associated to skew braces

[Rump, 2007] traced a novel research direction for finding solutions by introducing the algebraic structure of *brace*. Interesting generalizations have been produced over the years.

[Rump, 2007 - Guarnieri, Vendramin, 2017]

A triple $(B, +, \circ)$ is called *skew brace* if $(B, +)$ and $(B, \circ)$ are groups and it holds

$$\forall a, b, c \in B \qquad a \circ (b + c) = a \circ b - a + a \circ c.$$

If $(B, +)$ is abelian, then $(B, +, \circ)$ is a *brace*.

Any skew brace B gives rise to a *non-degenerate bijective solution*

$$r_B(a, b) = (-a + a \circ b, (-a + a \circ b)^{-} \circ a \circ b)$$

that is *involutive*, i.e., $r^2 = \mathrm{id}_{B\times B}$, if and only if $(B, +, \circ)$ is a brace.

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The opposite skew brace

If $(B; +; \circ)$ is a skew brace, one can consider the skew brace

$$B^{op} = (B; +^{op}; \circ)$$

with $a +^{op} b = b + a$, called the *opposite skew brace* of $(B; +; \circ)$.

As shown by **[Koch, Truman, 2020]**, considered the solution $r_{B^{op}}$ associated to the skew brace B^{op}

$$r_{B^{op}}(a; b) = (a \circ b - a; (a \circ b - a)^{-} \circ a \circ b);$$

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Semi-braces

[Catino, Colazzo, Stefanelli, 2017]

A *(left cancellative) semi-brace* is a triple $(S, +, \circ)$ such that $(S, +)$ is a left cancellative semigroup, $(S, \circ)$ is a group, and

$$\forall a, b, c \in S \qquad a \circ (b + c) = a \circ b + a \circ (a^- + c),$$

where a^- denotes the inverse of a with respect to $\circ$.

Every skew brace $(B, +, \circ)$ is a (left cancellative) semi-brace since

$$a \circ (a^- + c) = -a + a \circ c,$$

for all $a, c \in B$.

If $(S, +, \circ)$ is a (left cancellative) semi-brace, the map

$$r_S(a, b) = (a \circ (a^- + b), (a^- + b)^- \circ b)$$

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is a solution if and only if

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Inverse semi-braces

[Catino, M., Stefanelli, 2021]

An *inverse semi-brace* is a triple $(S, +, \circ)$ such that $(S, +)$ is a semigroup, $(S, \circ)$ is an inverse semigroup, and

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A semigroup $(S, \circ)$ is called *inverse* if, for each $a \in S$, there exists a unique $a^{-} \in S$ satisfying

$$a \circ a^{-} \circ a = a \quad \text{and} \quad a^{-} \circ a \circ a^{-} = a^{-}.$$

Every group is an inverse semigroup and the idempotent elements commute each other.

Under suitable conditions, the map r_S associated to an inverse semi-brace $(S, +, \circ)$ is a solution.

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Weak braces

Definition (Catino, M., Miccoli, Stefanelli, 2022)

A *weak brace* is a triple $(S, +, \circ)$ such that $(S, +)$ and $(S, \circ)$ are inverse semigroups satisfying

- $\forall a, b, c \in S \quad a \circ (b + c) = a \circ b - a + a \circ c,$
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Idempotents of weak braces

In any weak brace $E(S;+) = E(S;\circ)$ thus we will simply write $E(S)$.
As a consequence, if $|E(S)| = 1$, then S is a skew brace.

Key Lemma

Let $(S;+;\circ)$ be a weak brace. Then, it holds

$$e + a = e \circ a;$$

for all $e \in E(S)$ and $a \in S$.

In particular, $E(S)$ also is a (trivial) weak brace.

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for all $a, b \in S$, is a solution that has a *behaviour close to bijectivity*.

Given a weak brace $(S, +, \circ)$, we can consider its *opposite weak brace* that is $S^{op} = (S, +^{op}, \circ)$, with $a +^{op} b = b + a$, for all $a, b \in S$.

The solution $r_{S^{op}}$ associated to the opposite weak brace S^{op} of S is such that

$$r_S\, r_{S^{op}}\, r_S = r_S, \qquad r_{S^{op}}\, r_S\, r_{S^{op}} = r_{S^{op}}, \qquad \text{and} \qquad r_S r_{S^{op}} = r_{S^{op}} r_S.$$

Hence, r_S is a completely regular element of $\mathrm{Map}(S \times S)$.

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Structural properties of weak braces

Theorem

Let $(S, +, \circ)$ be a weak brace. Then, $(S, +)$ is a Clifford semigroup.

An inverse semigroup S is a *Clifford semigroup* if it has central idempotents.

Generally, $(S, \circ)$ is not a Clifford semigroup.

Example

Let $X := \{1, x, y\}$, S the upper semilattice on X with join 1, and T the commutative inverse monoid on X with identity 1 such that $xx = yy = x$ and $xy = y$. Consider the trivial weak braces on S and T. Let $:= (xy) \in \mathrm{Aut}(S)$ and $: T \to \mathrm{Aut}(S)$ defined by $(1) = (x) = \mathrm{id}_S$ and $(y) =$. Then, $S \ltimes T$ is a weak brace such that $(S \times T, \circ)$ is not Clifford, since

$$(y, y) \circ (y, y)^- = (y, x) \quad \& \quad (y, y)^- \circ (y, y) = (x, x).$$

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Weak braces coming from RB-operators

Examples of weak brace having can be obtained by using Rota-Baxter operators:

Definition (Catino, M., Stefanelli, 2023)

If $(S, +)$ is a Clifford semigroup, any map $\mathrm{R} : S \to S$ satisfying

$$\forall\, a, b \in S \qquad \mathrm{R}(a) + \mathrm{R}(b) = \mathrm{R}(a + \mathrm{R}(a) + b - \mathrm{R}(a))$$
$$a + \mathrm{R}(a) - \mathrm{R}(a) = a$$

is called *Rota–Baxter operator* on $(S, +)$.

Let R an RB-operator on a Clifford semigroup $(S, +)$. Set

$$\forall\, a, b \in S \qquad a \circ_{\mathrm{R}} b := a + \mathrm{R}(a) + b - \mathrm{R}(a),$$

then $(S, +, \circ_{\mathrm{R}})$ is a weak brace such that $(S, \circ_{\mathrm{R}})$ is a Clifford semigroup.

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Dual weak braces

Definition

A weak brace $(S, +, \circ)$ is called dual weak brace if $(S, \circ)$ is a Clifford semigroup.

If $(S, +, \circ)$ is a dual weak brace, the solution r_S has also a *behaviour close to the non-degeneracy* in the sense that

$$a \quad a^- \quad a = \quad a; \qquad a^- \quad a \quad a^- = \quad a^-; \quad \text{and} \quad a \quad a^- = \quad a^- \quad a$$

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Strong semilattice of skew braces

Let us consider the following:

- Y a (lower) semilattice;
- $\{B \quad Y\}$ a family of disjoint skew braces;
- For each pair ; of elements of Y such that ≥ , let ; $B \to B$ be a homomorphism of skew braces such that

  1: ; is the identical automorphism of B , for every Y;

  2: ; ; = ; if ≥ ≥

Then, $S = \{B \quad Y\}$ endowed with the operation given by

$$a \; B \; ; b \; B \qquad a + b = \; ; \; (a) + \; ; \; (b)$$
$$a \circ b = \; ; \; (a) \circ \; ; \; (b)$$

is a dual weak brace.

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Then, $S = \{B \quad \quad Y\}$ endowed with the operation given by

$$a \; B \; ; b \; B \qquad a + b = \;;\; (a) + \;;\; (b)$$
$$a \circ b = \;;\; (a) \circ \;;\; (b)$$

is a dual weak brace.

Strong semilattice of skew braces

Let us consider the following:

- Y a (lower) semilattice;
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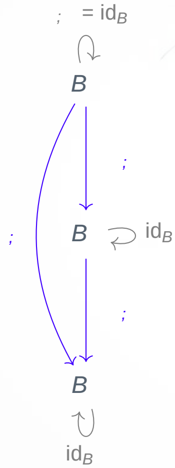

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Theorem (Catino, M., Stefanelli, 2023)

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Conversely, any dual weak brace is a strong semilattice of skew braces.

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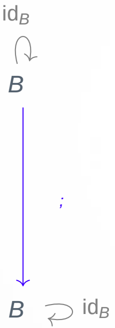
An easy example



Let us consider:

- $Y = \{\ ;\ \}$, with $>$,
- B the trivial skew brace on the cyclic group C_3,
- B the trivial skew brace on the symmetric group Sym_3
- $_{;}$ $C_3 \to \mathrm{Sym}_3$ the homomorphism given by $_{;}(0) = \mathrm{id}_3$, $_{;}(1) = (123)$, $_{;}(2) = (132)$:

Then, $S = B \ \ B$ endowed with the operation given by

$$a \ B\ ; b \ B \qquad a + b := \ _{;}(a) + \ _{;}(b)$$
$$a \circ b := \ _{;}(a) \circ \ _{;}(b)$$

is a (not trivial) dual weak brace.

Homomorphisms between skew braces

Problem

Finding homomorphism between skew braces for constructing dual weak braces.

This problem already emerged in literature, such as:

- **[Cedó, 2018], [Vendramin, 2019]** pose the problem of computing the automorphism groups of skew braces of size p^n
- **[Zenouz, 2019]** determines the automorphism group of skew braces of order $p > 3$
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Strong semilattice of solutions

Theorem (Catino, Colazzo, Stefanelli, 2021)

Let Y be a (lower) semilattice, $\{r \quad Y\}$ a family of disjoint solutions on X indexed by Y such that for each $\geq$ there is a map $_{;} \; X \to X$. Let $X = \{X \quad Y\}$ and $r \; X \times X \to X \times X$ be the map defined as

$$r(x;y) = r \quad (\quad _{;} \quad (x); \quad _{;} \quad (y));$$

for all $x \; X$ and $y \; X$. If the following conditions are satisfied:

1. $_{;}$ is the identity map of X, for every Y,
2. $_{;} \quad _{;} = \quad _{;}$, for all $; \; ; \; Y$ such that $\geq \; \geq$,
3. $(\; _{;} \times \; _{;}) r = r (\; _{;} \times \; _{;})$, for all $; \; Y$ such that $\geq$,

then r is a solution on X, called **strong semilattice of the solutions r**.

Strong semilattice of solutions

Theorem (Catino, Colazzo, Stefanelli, 2021)

Let Y be a (lower) semilattice, $\{r_\alpha \mid \alpha \in Y\}$ a family of disjoint solutions on X_α indexed by Y such that for each $\alpha \geq \beta$ there is a map $\varphi_{\alpha;\beta} : X_\alpha \to X_\beta$. Let $X = \bigcup \{X_\alpha \mid \alpha \in Y\}$ and $r : X \times X \longrightarrow X \times X$ be the map defined as

$$r(x;y) := r_{\alpha\beta}(\varphi_{\alpha;\alpha\beta}(x);\ \varphi_{\beta;\alpha\beta}(y));$$

for all $x \in X_\alpha$ and $y \in X_\beta$. If the following conditions are satisfied:

1. $\varphi_{\alpha;\alpha}$ is the identity map of X_α, for every $\alpha \in Y$,
2. $\varphi_{\beta;\gamma}\varphi_{\alpha;\beta} = \varphi_{\alpha;\gamma}$, for all $\alpha;\beta;\gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$,
3. $(\varphi_{\alpha;\beta} \times \varphi_{\alpha;\beta})\, r_\alpha = r_\beta\, (\varphi_{\alpha;\beta} \times \varphi_{\alpha;\beta})$, for all $\alpha;\beta \in Y$ such that $\alpha \geq \beta$,

then r is a solution on X, called strong semilattice of the solutions r_α.

The solutions coming from dual weak braces

Theorem

Let $S = [Y; B \;\; ; \;\;\; ; \;]$ be a dual weak brace. Then, the solution r associated to S is the strong semilattice of the bijective non-degenerate solutions r associated to each skew brace B .

Corollary

Let $S = [Y; B \;\; ; \;\;\; ; \;]$ be a *finite* dual weak brace and r the solution associated to S. Then, $r^{2k+1} = r$ with $2k = \mathrm{lcm}\{p(r\;)\;\;\;\; Y\}$.

Consequently, the solution r associated to a finite dual weak brace $S = [Y; B \;\; ; \;\;\; ; \;]$ is cubic, i.e., $r^3 = r$, if and only if r is involutive, i.e., $r^2 = \mathrm{id}_{B \;\; \times B}$, that is each B is a brace.

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Ideals of dual weak braces

Definition

A subset I of a dual weak brace S is an *ideal* of $(S, +, \circ)$ if

1. I is a normal subsemigroup of $(S, +)$,
2. $\lambda_a(I) \subseteq I$, for every $a \in S$,
3. I is a normal subsemigroup of $(S, \circ)$.

If I is an ideal, the relation $\sim_I$ on S given by

$$\forall\, a, b \in S \quad a \sim_I b \iff a - a = b - b \text{ and } -a + b \in I,$$

is a congruence of $(S, +, \circ)$.

Example

The set $\mathrm{Soc}(S) = \{a \mid a \in S, \ \forall\, b \in S \quad a + b = a \circ b \quad \text{and} \quad a + b = b + a\}$ is an ideal of any dual weak brace $(S, +, \circ)$ called *socle* of S.

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Let $S = [Y; B\ ;\ _{;}]$ be a dual weak brace, I an ideal of each skew brace B, and set $\ _{;} \square= \ _{;\ \square I}$, for all $\geq$.
If $\ _{;}(I\ ) \square I$, for any $>$, then $I = [Y; I\ ;\ _{;}]$ is an ideal of S and, conversely, every ideal of S is of this form.

Question

Let $S = [Y; B\ ;\ _{;}]$ be a dual weak brace.

$$\text{Is}\quad \mathrm{Soc}(S) = [Y; \mathrm{Soc}(B\ );\ _{;}]?$$

In general, the answer is negative. For instance, in the example with $Y = \{\ ;\ \}$, with $>$, $B = C_3$, and $B = \mathrm{Sym}_3$, we have that $\mathrm{Soc}(B\ ) \square \mathrm{Soc}(B\ )$ is not an ideal of S.

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Let $S = [Y; B \ ; \quad ;]$ be a dual weak brace, I an ideal of each skew brace B, and set $\quad ; \; := \quad ; \;$, for all $\geq$.
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Thank you!