The normal complement problem in group algebras

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Group Algebra

Group algebra

If R is a commutative ring, then RG is an algebra over R and is called group algebra.

- **Modular Group Algebra.** Let R be a commutative ring of characteristic p. If the group G has an element of order p, then RG is called modular group algebra.

- **Semisimple Group Algebra.** Let R be a commutative ring of characteristic p. If
  1. R is semisimple
  2. G is finite
  3. $|G|$ is invertible in R,

  then RG is called semisimple group algebra.

Normal Complement Problem

[1] H. N. Ward (1960)

For what p-groups G and field F containing p elements, there exists an epimorphism

$$\phi : \mathbb{U}(FG) \to G$$

fixing G. In other words, if there exist a normal subgroup N in $\mathbb{U}(FG)$, such that

$$\mathbb{U}(FG) = N \rtimes G?$$

[2]R. K. Dennis (1977)

What can be said in an arbitrary group ring RG.

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[1] H. N. Ward. Some results on the group algebra of a group over a prime field. In *Seminar on finite groups and related topics*, pages 13–19. Harvard University, 1960.

[2] R. Keith Dennis. The structure of the unit group of group rings. In *Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975)*, 103–130. Lecture Notes in Pure and Appl. Math., Vol. 26, 1977.

Connections

Isomorphism Problem

Whether a group algebra RG determines the group G ? In other words, does

$$RG \cong RH \implies G \cong H?$$

- If $R = \mathbb{Z}$ and G is finite nilpotent group then

$$(NCP) \implies (ISO).$$

- Also, if a normal complement of G in $\mathbb{U}(\mathbb{Z}G)$ is torsion free then

$$(NCP) \implies (ISO).$$

Connections

Fuchs' Problem

Which groups can be realized as the groups of units of a commutative ring.

- If G has a normal complement N in $\mathbb{U}(RG)$ such that $N-1$ is an ideal of RG, then

$$G \simeq \mathbb{U}\left(\frac{RG}{(N-1)}\right).$$

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Literature Survey

- The finite abelian p-groups and finite p-groups of exponent p with nilpotency class 2 have a normal complement in their corresponding unit groups over the field F_p.
[Moran and Tench, 1977]
- The dihedral groups, semi-dihedral groups and generalized quaternion groups of order 2^n, $n \geq 4$ do not have normal complement in their unit groups, over the field containing 2 elements. **[Ivory, 1980]**
- The existence of a torsion free normal complement of A_4 in $\mathbb{V}(\mathbb{Z}A_4)$ is shown. **[Allen and Hobby, 1980]**
- A normal complement of S_4 in $\mathbb{V}(\mathbb{Z}S_4)$ is analysed.
[Allen and Hobby, 1988]
- Any normal complement of A_4 in $\mathbb{V}(\mathbb{Z}A_4)$ is torsion free.
[Allen and Hobby, 1989]

Literature Survey

- A normal complement problem for central elementary by abelian p-groups over the field F_p has been explored. **[Sandling, 1989]**
- The problem is analysed for semisimple group algebras of finite cyclic groups and metacyclic groups of order $p_1.p_2$. **[Kaur et al., 2017]**
- The problem is investigated for modular group algebras of dihedral groups. **[Kaur and Khan, 2019]**

Symmetric and Alternating groups

Lemma[3]

Let

$$\mathbb{U}(FG) \overset{\phi}{\cong} H \times K$$

be a group isomorphism and

$$\pi : H \times K \to H$$

be projection map. If G has a normal complement in $\mathbb{U}(FG)$, then $\pi\phi(G)$ has a normal complement in H.

Theorem

Let FS_n denote the group algebra of S_n (n is even), over a finite field F of characteristic p, where $p > n$. Then S_n does not have normal complement in $\mathbb{U}(FS_n)$.

Theorem

There does not exist normal complement of A_n in $\mathbb{U}(FA_n)$ ($n \geq 4$), where F is finite field of characteristic $p > n$.

Theorem

Let FA_4 be the group algebra of A_4 over a finite field F of characteristic 3. Then A_4 does not have normal complement in $\mathbb{U}(FA_4)$.

[3] H. Setia and M. Khan. The normal complement problem in group algebras. *Communications in Algebra*, 50(1):287–291, 2022.

Field of characteristic 2

Theorem[4]

Let F be a finite field of characteristic 2.

1. If F contains 2 elements, then $1+\omega(FA_4)\omega(FK_4)$ is a normal complement to A_4 in $\mathbb{V}(FA_4)$.
2. If F contains $2^{2k},\ k \in \mathbb{N}$ or $2^{3r},\ r \in \mathbb{N}$ elements, then A_4 does not have a normal complement in $\mathbb{V}(FA_4)$.

- Class length of elements of $N \backslash 1+\Gamma(K_4)$ is computed with the help of GAP*.

[4] H. Setia and M. Khan. Normal complement problem over a finite field of characteristic 2. *Communications in Algebra*, 1–6, 2022.

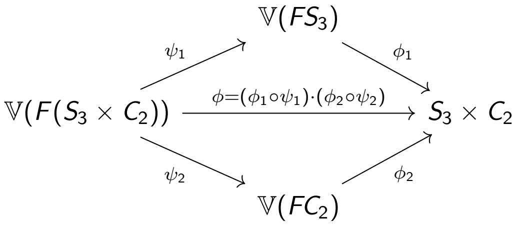
* Himanshu Setia and Manju Khan. GAP-code for calculating conjugacy class length. GitHub repository (2022). `https://github.com/HimanSetia/GAP-code-for-calculating-conjugacy-class-length.git`.

Field of characteristic 2

Proposition

Let F be a finite field of characteristic 2 and D_{4m} be the dihedral group of order $4m$, where m is an odd integer. Then, D_{4m} does not have a normal complement in $\mathbb{V}(FD_{4m})$, except for $m = 3$ and $|F| = 2$.

- $D_{12} = S_3 \times C_2$.



- N is an elementary abelian 2-group of order 2^6, when $m = 3$ and $|F| = 2$.

Field of characteristic 2

- $\mathbb{V}(FS_3) = \langle bu \rangle \rtimes S_3$, where $u = 1 + (1+b)a(1+b)$.
- $S_4 = \langle x, y \rangle \rtimes \langle a, b \rangle \cong K_4 \rtimes S_3$.

Proposition

Let F be the field containing 2 elements. Then, normal complement of S_4 in $\mathbb{V}(FS_4)$ is $(1 + \omega(FS_4)\omega(FK_4)) \rtimes \langle bu \rangle$.

Semisimple group algebras

- Let F_qG denote a semisimple group algebra and $\phi : \mathcal{U}(F_qG) \to \prod_{i=1}^{r} GL(n_i, q^{k_i})$ be an isomorphism.
- Let $\pi_i|_{\phi(G)}$ be the restriction map of π_i on $\phi(G)$ and K_i the kernel of $\pi_i|_{\phi(G)}$.

[5] Theorem

If n does not divide $(q^{k_i} - 1)|K_i|$ for some i, $1 \leq i \leq r$ then G does not have a normal complement in $\mathcal{U}(F_qG)$.

[5] H. Setia and M. Khan. A note on the normal complement problem in semisimple group algebras. (under review)

Simple and perfect groups

Corollary

Let G be a non-abelian simple group of order n. For every i, $1 \leq i \leq r$ with $n_i > 1$, if n does not divide $q^{k_i} - 1$ then G does not have normal complement in $\mathcal{U}(F_q G)$.

Definition

A group G is said to be perfect if $G = G^{'}$, where G' is the commutator subgroup of G. For example, A_5, $SL(2,5)$, etc.

- If G has a normal complement in $\mathbb{U}(FG)$, then $G^{'}$ has a normal complement in $\mathbb{U}^{'}(FG)$.

Theorem

Let G be a perfect group of order n. If n does not exceed $|SL(n_i, q^{k_i})|/d_i$ for some i, then G does not have a normal complement in $\mathcal{U}(F_q G)$. Here $d_i = \gcd(n_i, q^{k_i} - 1)$.

Corollary

SL(2,5) does not have a normal complement in $\mathcal{U}(F_q SL(2,5))$ *for* $q > 5$.

- $120 < \dfrac{|SL(6,q)|}{6} < \dfrac{|SL(6,q)|}{\gcd(6,q-1)}$.

Theorem

If $q > 3$ and n divides $1 + q^u$ for some integer $u \geq 1$, then D_{2n} does not have normal complement in $\mathbb{U}(F_q D_{2n})$.

Theorem

If $2n$ divides $1 + q^u$ for some integer $u \geq 1$, then Q_{4n} does not have normal complement in $\mathbb{U}(F_q Q_{4n})$.

Groups of order p^3

Let G be a non-abelian group of order p^3 for an odd prime p such that $\gcd(p, q) = 1$. It is known that

$$F_q G \cong F_q \bigoplus {F_{q^f}}^{((1+p)e)} \bigoplus M_p(F_{q^f})^{(e)},$$

where f denotes the order of q modulo p and $e = \frac{p-1}{f}$.

Theorem

If p^3 does not divide $q^f - 1$, then G does not have a normal complement in $\mathbb{U}(F_q G)$.

Allen, P. J. and Hobby, C. (1980).
A characterization of units in $V(ZA_4)$.
Journal of Algebra, 66(2):534–543.

Allen, P. J. and Hobby, C. (1988).
A characterization of units in $\mathcal{V}(\mathbb{Z}S_4)$.
Communications in Algebra, 16(7):1479–1505.

Allen, P. J. and Hobby, C. (1989).
Elements of finite order in $V(ZA_4)$.
Pacific Journal of Mathematics, 138(1):1 – 8.

Ivory, L. R. (1980).
A note on normal complements in mod p envelopes.
Proc. Amer. Math. Soc., 79(1):9–12.

Kaur, K., Khan, M., and Chatterjee, T. (2017).
A note on normal complement problem.
J. Algebra Appl., 16(1):1750011, 11.

Kaur, S. and Khan, M. (2019).
A note on normal complement problem for split metacyclic groups.
Communications in Algebra, 47(9):3842–3848.

Kaur, S. and Khan, M. (2020).
The normal complement problem and the structure of the unitary subgroup.
Communications in Algebra, 48(8):3628–3636.

Moran, L. E. and Tench, R. N. (1977).
Normal complements in mod p-envelopes.
Israel J. Math., 27(3-4):331–338.

Sandling, R. (1989).
The modular group algebra of a central-elementary-by-abelianp-group.
Archiv der Mathematik, 52(1):22–27.

Thank You for attending and listening to my presentation.