A family of set-theoretical solutions of the Yang-Baxter equation associated to a skew brace

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Joint work with
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Given a set B, a map $r : B \times B \to B \times B$ satisfying the *braid relation*

$$(r \times \mathrm{id}_B)(\mathrm{id}_B \times r)(r \times \mathrm{id}_B) = (\mathrm{id}_B \times r)(r \times \mathrm{id}_B)(\mathrm{id}_B \times r)$$

is said to be a *set-theoretical solution*, or briefly *solution*, of the YBE.

If we consider two maps $\lambda_a, \rho_b : B \to B$ and write r as

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

for all $a, b \in B$, then r is said to be

- *left non-degenerate* if λ_a is bijective, for every $a \in B$;
- *right non-degenerate* if ρ_b is bijective, for every $b \in B$;
- *non-degenerate* if r is both left and right non-degenerate.

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Some solutions on groups

Theorem (Lu, Yan, Zhu - 2000)

Let G be a group, $\lambda, \rho : G \to \mathrm{Sym}_G$ maps and set $\lambda_a(b) := \lambda(a)(b)$ and $\rho_b(a) := \rho(b)(a)$. If $\lambda, \rho : G \to \mathrm{Sym}_G$ are a left action and a right action of G on itself, respectively, and

$$\forall\ a, b \in G \quad a\, b = \lambda_a(b)\ \rho_b(a),$$

then the map $r : G \times G \to G \times G$ defined by

$$r(a, b) = (\lambda_a(b)\,,\ \rho_b(a))$$

is a non-degenerate bijective solution on G.

Venkov solutions

If G is a group and, for all $a, b \in G$, set $\lambda_a = \mathrm{id}_G$ and $\rho_b(a) = b^{-1}ab$, then λ and ρ satisfy *Lu, Yan, Zhu conditions* on G and the map $r : G \times G \to G \times G$ defined by

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Rump's approach

In 2007, Rump traced a novel research direction in the study of solutions.

Any Jacobson radical ring $(B, +, \cdot)$ determines a solution r on B that is the map $r : B \times B \to B \times B$ defined by

$$r(a, b) := \left(\lambda_a(b),\ \lambda^{-1}_{\lambda_a(b)}(a)\right)$$

where $\lambda_a(b) := a \cdot b + b$, for all $a, b \in B$. In particular, r is non-degenerate and involutive, i.e., $r^2 = \mathrm{id}_{B \times B}$.

More generally, non-degenerate involutive solutions are strictly related to the structure of *braces*. Even more generally, non-degenerate bijective solutions can be obtained through *skew braces*.

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Skew left braces

Definition (Rump - 2007; Guarnieri, Vendramin - 2017; Cedó, Jespers, and Okniński - 2014)

A triple $(B, +, \circ)$ is a *skew left brace* if $(B, +)$ and $(B, \circ)$ are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds, for all $a, b, c \in B$. If $(B, +)$ is abelian then B is a *left brace*.

The groups $(B, +)$ and $(B, \circ)$ have the same identity that we denote by 0.

- If $(B, +)$ is a group, then $(B, +, +)$ and $(B, +, +^{op})$ are skew left braces.
- Any Jacobson radical ring is a left brace. Indeed, if $(B, +, \cdot)$ is a Jacobson radical ring, then $(B, +, \circ)$ is a left brace with $\circ$ is the adjoint operation, where $\mathbf{a} \circ \mathbf{b} := \mathbf{a} + \mathbf{b} + \mathbf{a} \cdot \mathbf{b}$, for all $a, b \in B$.
- Any commutative left brace is a Jacobson radical ring. Indeed, if $(B, +, \circ)$ is a left brace such that $\circ$ is commutative, then $(B, +, \cdot)$ is Jacobson radical ring where $\mathbf{a} \cdot \mathbf{b} := \mathbf{a} \circ \mathbf{b} - \mathbf{a} - \mathbf{b}$, for all $a, b \in B$.

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Rump's solution associated to a skew brace

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If $(B, +, \circ)$ is a skew brace, then the map $r_B : B \times B \to B \times B$ defined by

$$r_B(a,\ b) := \left(-a + a \circ b,\ (-a + a \circ b)^{-} \circ a \circ b\right)$$

is a non-degenerate bijective solution (with a^{-} the inverse of a with respect to $\circ$, for every $a \in B$).

- r_B is involutive $\iff$ $(B, +, \circ)$ is a brace.
- Set, for all $a, b \in B$,

$$\lambda_a(b) := -a + a \circ b \quad \text{and} \quad \rho_b(a) := (-a + a \circ b)^{-} \circ a \circ b,$$

and consider $\lambda : B \to \mathrm{Sym}_B, a \mapsto \lambda_a$ and $\rho : B \to \mathrm{Sym}_B, a \mapsto \rho_b$. Then, λ and ρ satisfy *Lu, Yan, Zhu conditions* on $(B, \circ)$ since λ and ρ determine a left action and a right action of $(B, \circ)$ on itself, respectively, and

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Other solutions associated to skew left braces

[Doikou, Rybołowicz - 2022]: A new family of solutions can be obtained from any skew left brace B by *"deforming"* r_B by certain parameters.

Let $(B,+,\circ)$ be a skew left brace and $z \in B$ such that the identity

$$(a-b+c)\circ z = a\circ z - b\circ z + c\circ z \qquad (*)$$

holds, for all $a,b,c \in B$. Then, the map $\check{r}_z : B\times B \to B\times B$ given by

$$\check{r}_z(a,b) = \left(a\circ b - a\circ z + z,\ (a\circ b - a\circ z + z)^- \circ a\circ b\right),$$

is a non-degenerate and bijective solution, called *deformed solution by z on B*. Under the above assumption $(*)$, the map r_z also is

$$r_z(a,b) = \left(-a\circ z + a\circ b\circ z, (-a\circ z + a\circ b\circ z)^- \circ a\circ b\right)$$

a non-degenerate and bijective solution.

In particular, $\check{r}_z^{-1} = r_{z^-}$. Clearly, $r_0 = r_B$ and $\check{r}_0 = r_B^{-1}$.

First hint: Two-sided skew braces are crucial in the investigation of deformed solutions.

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$$\check{r}_z(a, b) = \left(a \circ b - a \circ z + z,\ (a \circ b - a \circ z + z)^{-} \circ a \circ b\right),$$

is a non-degenerate and bijective solution, called *deformed solution by z on B*. Under the above assumption $(*)$, the map r_z also is

$$r_z(a, b) = \left(-a \circ z + a \circ b \circ z, (-a \circ z + a \circ b \circ z)^{-} \circ a \circ b\right)$$

a non-degenerate and bijective solution.

In particular, $\check{r}_z^{-1} = r_{z^-}$. Clearly, $r_0 = r_B$ and $\check{r}_0 = r_B^{-1}$.

First hint: Two-sided skew braces are crucial in the investigation of deformed solutions.

First remarks on deformed solutions

If $(B, +, \circ)$ is a skew left brace, hereinafter the focus will be on the map

$$r_z(a, b) = \left(-a \circ z + a \circ b \circ z,\ (-a \circ z + a \circ b \circ z)^{-} \circ a \circ b\right)$$

and, for all $a, b \in B$, we fix the following notation:

$$\sigma_a^z(b) := -a \circ z + a \circ b \circ z \quad \text{and} \quad \tau_b^z(a) := (-a \circ z + a \circ b \circ z)^{-} \circ a \circ b.$$

If $z \in B$ gives rise to a deformed solution r_z, then the following hold:

- $\forall\, a, b \in B \quad a \circ b = \sigma_a^z(b) \circ \tau_b^z(a)$.
- the map $\tau^z : (B, \circ) \to \mathrm{Sym}_B, b \mapsto \tau_b^z$ is a group anti-homomorphism.
- Considered the map $\sigma^z : (B, \circ) \to \mathrm{Sym}_B, a \mapsto \sigma_a^z$, it holds that

$$\sigma^z \text{ is a homomorphism} \iff \forall\, a \in B \quad a \circ z = z + a$$

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The study of parameters

Question: What elements z in a skew left brace B ensure that r_z is a solution?

Definition (MRS - 2023)

Let $(B, +, \circ)$ be a skew left brace. Then, we call the set

$$\mathcal{D}_r(B) = \{z \in B \mid \forall\, a, b \in B \quad (a + b) \circ z = a \circ z - z + b \circ z\},$$

the *right distributor* of B.

Clearly, $0 \in \mathcal{D}_r(B)$.

A skew left brace $(B, +, \circ)$ is *two-sided*, i.e., the identity

$$(a + b) \circ c = a \circ c - c + b \circ c$$

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Theorem (MRS - 2023)

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Some examples

The other limit case is when there exists only the trivial deformation, namely, $\mathcal{D}_r(B) = \{0\}$.

Example 1

Let $B := (\mathbb{Z}, +, \circ)$ be the left brace on $(\mathbb{Z}, +)$ with $a \circ b = a + (-1)^a b$, for all $a, b \in \mathbb{Z}$ (cf. [Rump - 2007]). Then, $\mathcal{D}_r(B) = \{0\}$.

The following is an example of a skew left brace in which $\mathcal{D}_r(B)$ is not trivial.

Example 2

Let B be the left brace in Example 1 and $U_9 := \left(U\left(\mathbb{Z}/2^9\mathbb{Z}\right), +_1, \circ\right)$ the left brace where

$$\forall\ a, b \in U(\mathbb{Z}/2^9\mathbb{Z}) \quad a +_1 b = a - 1 + b \quad \text{and} \quad a \circ b = ab,$$

with $+$ and $\cdot$ the usual operations in the ring modulo 2^9. Then, $U_9 \times B$ is a left brace such that

$$\mathcal{D}_r(U_9 \times B) = U_9 \times \{0\}.$$

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Properties of the right distributor

Let $(B, +, \circ)$ be a skew left brace.

$$Z(B, \circ) \leq (\mathcal{D}_r(B), \circ) \leq (B, \circ)$$

In general, $(\mathcal{D}_r(B), +) \not\leq (B, +)$, unless we get into particular cases.

If $(B, +, \circ)$ is a left brace, then $\mathcal{D}_r(B)$ is a two-sided subbrace of B.

- $\mathrm{Fix}(B) \subseteq \mathcal{D}_r(B)$, where $\mathrm{Fix}(B) = \{a \in B \mid \forall\, x \in B\ \lambda_x(a) = a\}$.
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Is $\mathcal{D}_r(B)$ an ideal of B?

It becomes natural to wonder when $\mathcal{D}_r(B)$ is an ideal of a skew left brace $(B, +, \circ)$.

Let us recall that a subset I of B is an *ideal* of B if it is both a normal subgroup of $(B, +)$ and $(B, \circ)$ and I is *λ-invariant*, namely $\lambda_a(I) \subseteq I$, for every $a \in B$.

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Let $U_9 \times B$ the left brace seen before. Then, the right distributor $\mathcal{D}_r(U_9 \times B)$ is an ideal of $U_9 \times B$.

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Important remark on deformed solutions

Left braces may determine non-involutive solutions.

Example [Doikou, Rybołowicz - 2022]

Consider $\text{Odd} := \left\{ \frac{2n+1}{2k+1} \mid n, k \in \mathbb{Z} \right\}$ and the structure of brace $(\text{Odd}, +_1, \circ)$ where the binary operation $+_1$ and $\circ$ are given by

$$\forall\, a, b \in \text{Odd} \qquad a +_1 b := a - 1 + b \qquad \text{and} \qquad a \circ b := a \cdot b$$

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Equivalent or not equivalent solutions?

Question: Let $(B,+,\circ)$ be a skew left brace. For which parameters $z,w \in B$, are the deformed solutions r_z and r_w equivalent?

[Etingof, Schedler, Soloviev - 1999]: Two solutions r and s on two sets X and Y, respectively, are said to be *equivalent* if there exists a bijective map $\varphi : X \to Y$ such that

$$(\varphi \times \varphi)\, r = s\, (\varphi \times \varphi)\,,$$

namely, the diagram

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The two-sided case

If r_z and r_w are two deformed solutions and $\varphi \in \mathrm{Aut}(B, +, \circ)$ such that $\varphi(z) = w$, then r_z and r_w are trivially equivalent via φ.

In the special case of a two-sided skew brace, such a map φ exists when z and w are in the same conjugacy class.

Proposition (MRS - 2023)

Let $(B, +, \circ)$ be a two-sided skew brace and $z, w \in B$ belonging to the same conjugacy class in $(B, \circ)$. Then, the deformed solutions r_z and r_w are equivalent.

[Nasybullov - 2019; Trappeniers - 2023]: All the inner automorphisms of $(B, \circ)$ are skew brace automorphisms of B.

The converse is not true.

Consider the trivial left brace $(B, +, +)$ on the cyclic group $\mathbb{Z}/2\mathbb{Z}$. Then, the solutions r_0 and r_1 coincide, but 0 and 1 trivially belong to different conjugacy classes.

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Let $(B, +, \circ)$ be a two-sided skew brace and $z, w \in B$ belonging to the same conjugacy class in $(B, \circ)$. Then, the deformed solutions r_z and r_w are equivalent.

[Nasybullov - 2019; Trappeniers - 2023]: All the inner automorphisms of $(B, \circ)$ are skew brace automorphisms of B.

The converse is not true.

Consider the trivial left brace $(B, +, +)$ on the cyclic group $\mathbb{Z}/2\mathbb{Z}$. Then, the solutions r_0 and r_1 coincide, but 0 and 1 trivially belong to different conjugacy classes.

The two-sided case

If r_z and r_w are two deformed solutions and $\varphi \in \mathrm{Aut}\,(B,+,\circ)$ such that $\varphi\,(z) = w$, then r_z and r_w are trivially equivalent via φ.

In the special case of a two-sided skew brace, such a map φ exists when z and w are in the same conjugacy class.

Proposition (MRS - 2023)

Let $(B,+,\circ)$ be a two-sided skew brace and $z, w \in B$ belonging to the same conjugacy class in $(B,\circ)$. Then, the deformed solutions r_z and r_w are equivalent.

[Nasybullov - 2019; Trappeniers - 2023]: All the inner automorphisms of $(B,\circ)$ are skew brace automorphisms of B.

The converse is not true.
Consider the trivial left brace $(B,+,+)$ on the cyclic group $\mathbb{Z}/2\mathbb{Z}$. Then, the solutions r_0 and r_1 coincide, but 0 and 1 trivially belong to different conjugacy classes.

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Thank you!

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