Skew bracoids and quotients of skew braces

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Groups, rings, and the Yang-Baxter equation

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Overview

- Joint with Isabel Martin-Lyons (Keele) and Ilaria Colazzo (Exeter)

Aim

Define a new algebraic object, motivated by certain quotients of skew braces, and explore connections with Hopf-Galois structures and the Yang-Baxter equation.

- Skew braces, ideals, and quotients
- Skew bracoids
- Characterizations and substructures
- Connections with Hopf-Galois strutures
- Connections with the Yang-Baxter equation

Skew braces, ideals, and quotients

Definition

A *skew (left) brace* is a triple $(G, \star, \circ)$ in which $(G, \star)$ and $(G, \circ)$ are groups and

$$x \circ (y \star z) = (x \circ y) \star x^{-1} \star (x \circ z) \text{ for all } x, y, z \in G,$$

where x^{-1} denotes the inverse of x with respect to $\star$.

Let $(G, \star, \circ)$ be a skew brace. Then

- there is a homomorphism $\lambda : (G, \circ) \to \mathrm{Aut}(G, \star)$ defined by
$$\lambda_x(y) = x^{-1} \star (x \circ y);$$
- there is an antihomomorphism $\rho : (G, \circ) \to \mathrm{Perm}(G)$ defined by
$$\lambda_x(y) \circ \rho_y(x) = x \circ y;$$
- the function $r : G \times G \to G \times G$ defined by $r(x, y) = (\lambda_x(y), \rho_y(x))$ is a bijective nondegenerate solution of the Yang-Baxter equation.

Skew braces, ideals, and quotients

Let $(G, \star, \circ)$ be a skew brace.

Definition

A subgroup H of $(G, \star)$ is called

- a *left ideal* if $\lambda_x(y) \in H$ for all $x \in G$ and $y \in H$;
- a *strong left ideal* if it is a left ideal and is normal in $(G, \star)$;
- an *ideal* if it is a strong left ideal and is normal in $(G, \circ)$.

Proposition

If H is an ideal of $(G, \star, \circ)$ then $(G/H, \star, \circ)$ is a skew brace.

Skew braces, ideals, and quotients

Question

What if we try to quotient by a strong left ideal H of $(G, \star, \circ)$?

- For all $x \in G$ we have $x \star H = x \circ H = xH$, say.
- The coset space G/H is a quotient group with respect to $\star$, but not with respect to $\circ$
- The group $(G, \circ)$ acts transitively on G/H by $x \odot (yH) = (x \circ y)H$.
- We have

$$\begin{aligned} x \odot (yH \star zH) &= (x \circ (y \star z))H \\ &= ((x \circ y) \star x^{-1} \star (x \circ z))H \\ &= (x \odot yH) \star (x \odot eH)^{-1} \star (x \odot zH). \end{aligned}$$

Skew bracoids

Definition

A *skew bracoid* is a 5-tuple $(G, \circ, N, \star, \odot)$ where $(G, \circ)$ and $(N, \star)$ are groups and $\odot$ is a transitive action of $(G, \circ)$ on N such that

$$g \odot (\eta \star \mu) = (g \odot \eta) \star (g \odot e_N)^{-1} \star (g \odot \mu)$$

for all $g \in G$ and $\eta, \mu \in N$.

- Where possible, we write simply $(G, N, \odot)$, or even (G, N).
- For now, we always assume G, N are finite. Then $|G| = |S||N|$, where $S = \mathrm{Stab}_G(e_N)$.
- Every skew brace is a skew bracoid, with $\odot$ and $\circ$ coinciding.
- If $|N| = |G|$ then (G, N) is essentially a skew brace.

Some characterizations

Theorem

Let $(G, \circ), (N, \star)$ be groups. The following are equivalent:

1. *A transitive action $\odot$ of G on N such that $(G, \circ, N, \star, \odot)$ is a skew bracoid;*
2. *a transitive subgroup A of $\mathrm{Hol}(N) = N \rtimes \mathrm{Aut}(N)$ isomorphic to a quotient of G;*
3. *a homomorphism $\lambda : G \to \mathrm{Aut}(N)$ and a surjective 1-cocycle $\pi : G \twoheadrightarrow N$.*

- The implication (1) $\to$ (2) uses the permutation representation $\mathcal{L}_\odot : G \to \mathrm{Perm}(N)$
- The implication (2) $\to$ (3) gives rise to the λ-function of a skew bracoid.

Equivalence

Definition

Two skew bracoids (G, N) and (G', N') are called *equivalent* if $N = N'$ and $\mathcal{L}_{\odot}(G) = \mathcal{L}_{\odot'}(G') \subseteq \mathrm{Hol}(N)$.

- The analogous notion for skew braces is "equal".

Proposition

Let $(N, \star)$ be a group. There is a bijective correspondence between transitive subgroups of $\mathrm{Hol}(N)$ and equivalence classes of skew bracoids (G, N).

λ-functions and ideals

Proposition

Let (G, N) be a skew bracoid. There is a homomorphism $\lambda : G \to \mathrm{Aut}(N)$ defined by

$$\lambda_g(\eta) = (g \odot e_N)^{-1} \star (g \odot \eta).$$

Definition

A *left ideal* of a skew bracoid (G, N) is a subgroup M of N such that $\lambda_g(M) = M$ for all $g \in G$. An *ideal* is a left ideal M that is normal in N.

Proposition

If M is an ideal of (G, N) then $(G, N/M)$ is a skew bracoid.

Connections with Hopf-Galois structures

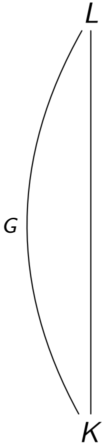
- A *Hopf-Galois structure* on a finite extension of fields L/K consists of
  - a K-Hopf algebra $\mathcal{H}$ and
  - a K-linear action of $\mathcal{H}$ on L satisfying a certain nondegeneracy condition.
- A given extension may admit numerous different Hopf-Galois structures.
- If $\mathcal{H}$ gives a Hopf-Galois structure on L/K then each Hopf subalgebra $\mathcal{H}'$ of $\mathcal{H}$ yields a "fixed field" $L^{\mathcal{H}'}$.
- The resulting "Hopf-Galois correspondence" is injective and inclusion reversing, but not surjective in general.
- We say that the intermediate field $L^{\mathcal{H}'}$ is *realizable with respect to* $\mathcal{H}$.

Connections with Hopf-Galois structures

In the case that L/K is a Galois extension with Galois group $(G, \circ)$, Stefanello and Trappeniers show that there is a bijection between

- binary operations $\star$ on G such that $(G, \star, \circ)$ is a skew brace;
- Hopf-Galois structures on L/K,

Furthermore, for each Hopf-Galois structure there is a bijection between the realizable intermediate fields and the left ideals of the corresponding skew brace.

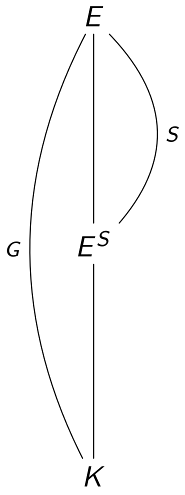


Connections with Hopf-Galois structures

Theorem

Let E/K be a finite Galois extension with Galois group $(G, \circ)$, and let $S \leq G$. There is a bijection between

- *binary operations $\star$ on $X = G/S$ such that $(G, \circ, X, \star, \odot)$ is a skew bracoid;*
- *Hopf-Galois structures on E^S/K.*



Furthermore, for each Hopf-Galois structure there is a bijection between the realizable intermediate fields and the left ideals of the corresponding skew bracoid.

Connections with the Yang-Baxter equation

Let $(G, \star, \circ)$ be a skew brace, and suppose that there exists a strong left ideal H and a subskew brace C such that

- $(G, \star) = H \rtimes C$ and
- $(G, \circ) = H \circ C$.

Consider the skew bracoid $(G, \circ, G/H, \star, \odot)$ and the homomorphism

$$\lambda : G \to \mathrm{Aut}(G/H, \star).$$

We obtain a homomorphism

$$\widehat{\lambda} : G \to \mathrm{Hom}_{\star}(G, C).$$

Define $\widehat{\rho} : G \to \mathrm{Perm}(G)$ by $\widehat{\lambda}_x(y) \circ \widehat{\rho}_y(x) = x \circ y$.

Theorem

The function $r : G \times G \to G \times G$ defined by $r(x, y) = (\widehat{\lambda}_x(y), \widehat{\rho}_y(x))$ is a bijective left degenerate solution of the Yang-Baxter equation.

Some natural questions

Question

Do skew bracoids have anything to do with groupoids?

- We think so, but we're not sure whether this perspective is beneficial.

Question

What are some other applications of skew bracoids?

- We think they will have applications to classifying skew braces: e.g. via short exact sequences.

Question

Does every skew bracoid occur as the quotient of a skew brace by a strong left ideal?

- We don't know!

Thank you for your attention.

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