CENTRALLY NILPOTENT SKEW BRACES

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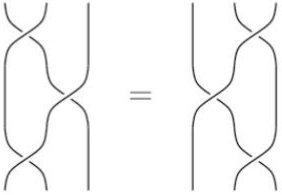
THE YANG-BAXTER EQUATION: A PICTURE

Definition

A set-theoretic solution to the Yang-Baxter equation is a tuple (X, r), where X is a set and $r : X \times X \longrightarrow X \times X$ a function such that (on X^3)

$$(r \times \mathrm{id}_X)\,(\mathrm{id}_X \times r)\,(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)\,(r \times \mathrm{id}_X)\,(\mathrm{id}_X \times r)\,.$$

For further reference, denote $r(x, y) = (\lambda_x(y), \rho_y(x))$.



DEFINITIONS AND EXAMPLES

Definition

A set-theoretic solution (X, r) is called

- left (resp. right) non-degenerate, if λ_x (resp. ρ_y) is bijective,
- non-degenerate, if it is both left and right non-degenerate,
- involutive, if $r^2 = \mathrm{id}_{X \times X}$,

Examples

- Twist solution: $r(x, y) = (y, x)$,
- Lyubashenko, where $f, g : X \to X$ are maps with $fg = gf$: $r(x, y) = (f(y), g(x))$.

THE STRUCTURE MONOID AND GROUP

Definition

Let (X, r) be a set-theoretic solution of the Yang-Baxter equation. Then the group

$$G(X, r) = \left\langle x \in X \mid xy = \lambda_x(y)\rho_y(x) \right\rangle,$$

is called the structure group of (X, r).

WHAT ARE SKEW LEFT BRACES

Definition (Rump, CJO, GV)

Two groups $(A, +)$ and $(A, \circ)$ form a skew left brace $(A, +, \circ)$, if for any $a, b, c \in A$, it holds that

$$a \circ (b + c) = (a \circ b) - a + (a \circ c),$$

where $-a$ denotes the inverse of a in $(A, +)$.
Moreover, if $(A, +)$ is abelian, then $(A, +, \circ)$ is a left brace

EXAMPLES OF SKEW BRACES

Example

1. Every group $(G, +)$ has the skew left brace structure $(G, +, +)$, these are *trivial skew left braces*.
2. *The dihedral group* $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$ *has a left brace structure, where* $a^i b^j + a^k b^l = a^{i+k+jl} b^{j+l}$ *with* $j, l \in \{0, 1\}$.
3. *Radical rings.*

CREATING SOLUTIONS ON SKEW BRACES (1)

Definition (Rump, CJO, GV)

Let $(B,+)$ and $(B,\circ)$ be groups on the same set B such that for any $a,b,c \in B$ it holds that

$$a \circ (b+c) = (a \circ b) - a + (a \circ c).$$

Then $(B,+,\circ)$ is called a skew (left) brace

If $(B,+)$ is abelian, one says that $(B,+,\circ)$ is a left brace.

CREATING SOLUTIONS ON SKEW BRACES (1)

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Let $(B, +)$ and $(B, \circ)$ be groups on the same set B such that for any $a, b, c \in B$ it holds that

$$a \circ (b + c) = (a \circ b) - a + (a \circ c).$$

Then $(B, +, \circ)$ is called a skew (left) brace

If $(B, +)$ is abelian, one says that $(B, +, \circ)$ is a left brace.

Denote for $a, b \in B$, the map $\lambda_a(b) = -a + a \circ b$. Then, $\lambda : (B, \circ) \longrightarrow \mathrm{Aut}(B, +) : a \mapsto \lambda_a$ is a well-defined group morphism.

CREATING SOLUTIONS ON SKEW BRACES (2)

Theorem

Let $(B, +, \circ)$ be a skew left brace. Denote for any $a, b \in B$, the map $r_B(a, b) = (\lambda_a(b), \overline{(\overline{a} + b)} \circ b)$. Then (B, r_B) is a bijective non-degenerate solution. Moreover, if $(B, +)$ is abelian, then (B, r_B) is involutive.

Remark

Let (X, r) be a bijective non-degenerate set-theoretic solution. Then, $G(X, r)$ is a skew left brace and carries an associated solution as a skew brace.

THE *-OPERATION IN SKEW LEFT BRACES

Definition

Let $(A, +, \circ)$ be a skew left brace. For any $a, b \in A$, denote

$$a * b = -a + a \circ b - b = \lambda_a(b) - b.$$

Denote $X * Y$ for the additive subgroup generated by $x * y$, where $x \in X, y \in Y$ and $X, Y \subseteq A$.

Example

1. For $(G, +, +)$, one sees that $a * b = 0$. Actually a characterization.
2. For $(D_{2n}, +, \cdot)$ one can see that $(a^i b^j) * (a^k b^l) \in \langle a \rangle$.

SOLUTIONS LIKE LYUBASHENKO'S

Definition (Retraction)

Let (X, r) be a finite bijective non-degenerate set-theoretic solution. Define the relation $x \sim y$ on X, when $\lambda_x = \lambda_y$ and $\rho_x = \rho_y$. Then, there exists a natural set-theoretic solution on $X/\sim$ called the retraction $\mathrm{Ret}(X, r)$.

SOLUTIONS LIKE LYUBASHENKO'S

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Denote for $n \geq 2$, $\mathrm{Ret}^n(X, r) = \mathrm{Ret}\left(\mathrm{Ret}^{n-1}(X, r)\right)$. If there exists a positive integer n such that $|\mathrm{Ret}^n(X, r)| = 1$, then (X, r) is called a multipermutation solution

WHY ARE MULTIPERMUTATION SOLUTIONS INTERESTING

Theorem (CJOBVAGI)

Let (X, r) be a finite involutive non-degenerate set-theoretic solution. The following statements are equivalent,

- *the solution (X, r) is a multipermutation solution,*
- *the group $G(X, r)$ is left orderable,*
- *the group $G(X, r)$ is diffuse,*
- *the group $G(X, r)$ is poly-$\mathbb{Z}$.*

Breaks down for non-involutive solutions, as $G(X, r)$ has torsion in that case!

ALL MULTIPERMUTATION SOLUTIONS

Proposition

Let (X, r) be a multipermutation solution, then the skew brace $G(X, r)$ is of nilpotent type.

So we focus attention on so-called skew braces $(B, +, \circ)$ of **nilpotent type**, i.e. $(B, +)$ is a nilpotent group.

NON-ASSOCIATIVE, SO SIDES MATTER

Left Nilpotent

- $B^{n+1} = B * B^n$ left ideals
- $|B^k| = 1$, then left nilpotent
- Nilpotent type: $(B, \circ)$ nilpotent
- Example: (C_{2^n}, D_{2^n})

Right nilpotent

- $B^{(n+1)} = B^{(n)} * B$ ideals
- $|B^{(k)}| = 1$, then right nilpotent
- Nilpotent type: (B, r_B) multipermutation
- Example: (C_{2n}, D_{2n})

MEASURING MULTIPERMUTATION

Definition

A skew brace $(B, +, \circ))$ is said to be multipermutation, if (B, r_B) is multipermutation.

Equivalently:

- B is right nilpotent of nilpotent type,
- The chain $Soc^n(B)$ ends in B.

Here, $Soc^{n+1}(B)$ is the pullback in B of $Soc(B/Soc^n(B))$ with

$$Soc(A) = \ker \lambda \cap Z(B, +).$$

CENTRAL NILPOTENCY

Definition

Let B be a skew brace. Denote $Ann(B) = Soc(B) \cap Z(B, \circ)$.

Equivalently,

$$Ann(B) = \left\{x \in Z(B, +) \mid \lambda_x = \mathrm{id}_B, \lambda_y(x) = x \text{ for all } y \in B\right\}.$$

Definition

Let B be a skew brace. One says that B is centrally nilpotent, if the chain $Ann^n(B)$ ends in B, where $Ann^{k+1}(B)$ is pullback of $Ann(B/Ann^k(B))$.

DESCENDING SERIES

We have an ascending ideal series, what about descending?

$$\Gamma_{n+1}(B,I) = \langle B * \Gamma_n(B,I), \Gamma_n(B,I) * B, [\Gamma_n(B,I), B]_+ \rangle$$

is an ideal in B, if I is an ideal.

Proposition (Bonatto,Jedlicka)

Let B be a skew brace. Then, B is centrally nilpotent, if for some positive integer n we have $\Gamma_n(B,B) = 1$.

STRONGLY NILPOTENT

$$B^{[n]} = \left\langle B^{[i]} * B^{[n-i]} \mid 1 \le i \le n \right\rangle .$$

Proposition (Smoktunowicz)

Let B be a skew brace. Then, B is strongly nilpotent if and only if B is left and right nilpotent and $(B, \circ)$ is nilpotent.

What if we account for additive commutator?

$$\Gamma_{[n]}(B) = \left\langle \Gamma_{[i]}(B) * \Gamma_{[n-i]}(B), \left[\Gamma_{[i]}(B), \Gamma_{[n-i]}(B)\right]_{+} \right\rangle$$

Proposition (Jespers, AVA, Vendramin)

Let B be a skew brace of nilpotent type. If B is centrally nilpotent, then B is strongly centrally nilpotent. Moreover, B is strongly nilpotent.

NILPOTENCY CLASS

Both the chains $\Gamma_n(B)$ and $\Gamma_{[n]}(B)$ allow to define a notion of nilpotency class of B.

Problem

- *Can we relate the above nilpotency classes?*
- *Are there bounds using the additive/multiplicative nilpotency class?*

FINITELY GENERATED

Proposition (Jespers, AVA, Vendramin)

Let B be a centrally nilpotent skew brace with ACC on sub skew braces. TFAE

- *B is finitely generated as a brace,*
- $(B, +)$ *is finitely generated as a group,*
- $(B, \circ)$ *is finitely generated as a group.*

Vice versa, every finitely generated Centrally nilpotent skew brace has ACC on sub skew braces.

TORSION

What is torsion in a skew brace?

Proposition (Jespers, AVA, Vendramin)

Let B be a centrally nilpotent skew brace. Then $T_+(B) = T_\circ(B)$, which is an ideal of B. Finite, if B is finitely generated.

Proposition (Jespers, AVA, Vendramin)

Let B be a centrally nilpotent skew brace. If $T_+(B) = 0$. Then, $a^n = b^n$ or $na = nb$ implies $a = b$.

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