# Duality structures on tensor categories coming from vertex operator algebras

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# Overview

1. Vertex (operator) algebras and commutative algebras
2. Tensor products
3. Duality structures

## Definition: Vertex algebra (VA)

Data:

- $\mathbb{C}$ vector space $V$
- field map $Y_z : V \otimes V \to V(\!(z)\!)$
- vacuum vector $\Omega \in V$.

Axioms:

- vacuum axiom: $Y_z(\Omega \otimes -) = \mathrm{id}_V$ and $Y_z(A \otimes \Omega) = A + zV[\![z]\!], \ \forall A \in V$
- locality: $\forall A, B \in V$, $\exists n \in \mathbb{N}$ $(z-w)^n [Y_z(A \otimes -), Y_w(B \otimes -)] = 0$

## Consequence/Proposition

- There exists a translation operator $T : V \to V$, such that $T\Omega = 0$ and $[T, Y_z] = \partial_z Y_z$
- For all $A, B, C \in V$

$$Y_z(A \otimes Y_w(B \otimes C)) \in V(\!(z)\!)(\!(w)\!)$$
$$Y_w(B \otimes Y_z(A \otimes C)) \in V(\!(w)\!)(\!(z)\!)$$
$$Y_w(Y_{z-w}(A \otimes B) \otimes C) \in V(\!(w)\!)(\!(z-w)\!)$$

are expansions of same element in $V[\![z,w]\!][z^{-1}, w^{-1}, (z-w)^{-1}]$.

# Example: Heisenberg vertex algebra

- Heisenberg Lie algebra: $\mathfrak{h} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}a_n \oplus \mathbb{C}\mathbf{1}$ with $\mathbf{1}$ central and relations $[a_m, a_n] = m\delta_{m,n}\mathbf{1}$.
- Fock space ($\mathfrak{h}$ Verma module), $F_\lambda = \mathbb{C}[a_{-n} : n \geq 1]\,|\lambda\rangle$, where $a_0 = \lambda\mathrm{id},\ a_n/n = \partial_{a_{-n}}, n \geq 1$.
- The assignments and recursions

  $|0\rangle \otimes - \mapsto \mathrm{id}_{F_0}, \quad a_{-1}\,|0\rangle \otimes - \mapsto \sum_{i\in\mathbb{Z}} a_i z^{-i-1} = a(z)$,

  $a_{-n}\,|0\rangle \otimes - \mapsto \frac{\partial^{n-1}}{(n-1)!}a(z) \otimes -$,

  $a_{-n}v \otimes - \mapsto \left(\frac{\partial^{n-1}}{(n-1)!}a(z)\right)_{\mathsf{r}} Y_z(v \otimes -) + Y_z\left(v \otimes \left(\frac{\partial^{n-1}}{(n-1)!}a(z)\right)_{\mathsf{p}} -\right)$.

  define a vertex algebra on $F_0$ with vacuum vector $|0\rangle$.

## Definition: Vertex operator algebra (VOA)/ conformal vertex algebra

A vertex operator algebra is a vertex algebra $(V, Y, \Omega, T)$ admiting a conformal vector $\omega \in V$ satsifying

- Virasoro algebra relations: $Y_z(\omega \otimes -) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$

  $$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\ ,$$

  with $c \in \mathbb{C}$, central charge.
- Non-negative integral conformal grading:

  $$V = \bigoplus_{n=0}^{\infty} V_n,\ V_n = \{v \in V | (L_0 - n)v = 0\}.$$

- Conformal derivation: $L_{-1} = T$.

# Example: Heisenberg vertex operator algebra

Consider the vertex algebra $(F_0, Y)$ from before.

- For $\rho \in \mathbb{C}$, the vector $\omega_\rho = \left(\frac{1}{2}a_{-1}^2 + \rho a_{-2}\right)|0\rangle \mapsto \frac{1}{2}\left(a(z)_{\mathsf{r}}a(z) + a(Z)a(z)_{\mathsf{p}}\right) + \rho\partial a(z)$ is conformal.
- The central charge of the resulting Virasoro algebra is $c_\rho = 1 - 12\rho^2$.
- The conformal grading on $F_0$ assigns grade $0$ to $|0\rangle$ and grade $-n$ to $a_n$.

# Recap

- Vertex algebras are essentially associative commutative unital $\mathbb{C}$-algebras with a derivation.
- Fields $Y_z : V \otimes V \to V(\!(z)\!)$ are essentially an action of $V$ on itself.
- Where you have commutativie algebras, you have modules!

## Definition: Vertex algebra module

Let $(V, \Omega, T, Y)$ be a vertex algebra. A $V$-module is a pair $(M, Y^M)$: $M$ a vector space and $Y_z^M : V \otimes M \to M(\!(z)\!)$ a $V$-action, that is,

- $Y_z^M(\Omega \otimes -) = \mathrm{id}_M$
- For all $A, B \in V$ and $C \in M$ the expansions
  $Y_z^M(A \otimes Y_w^M(B \otimes C)) \in M(\!(z)\!)(\!(w)\!)$
  $Y_w^M(B \otimes Y_z^M(A \otimes C)) \in M(\!(w)\!)(\!(z)\!)$
  $Y_w^M(Y_{z-w}(A \otimes B) \otimes C) \in M(\!(w)\!)(\!(z-w)\!)$
  can be identified in $M[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$.

Many additional assumtions can be added to the above definition. E.g. bounded conformal weights, finite weight spaces, semi simplicity, etc.

# Example: Heisenberg modules

- Recall the Heisenberg algebra $\mathfrak{h}$ and Fock spaces $F_\lambda$ from before.
- For $\lambda \in \mathbb{C}$, the Fock space $F_\lambda$ is a $(F_0, Y)$-module with the action defined by the same formula, e.g. $Y_z^{F_\lambda}(a_{-1}\left|0\right\rangle \otimes \left|\lambda\right\rangle) = a(z)\left|\lambda\right\rangle$.

# Duals of modules

## Definition: the dual of a module

Let $(V, Y, \Omega, \omega)$ be a vertex operator algebra and $(M, Y^M)$, $M = \bigoplus_n M_n$ a module. Then $(M', Y^{M'})$

- $M' = \bigoplus_n \mathrm{Hom}_{\mathbb{C}}(M_n, \mathbb{C})$,
- $\langle Y_z^{M'}(v \otimes \mu), m \rangle = \langle \mu, Y_{z^{-1}}^M (e^{zL_1}(-z^2)^{L_0} v \otimes m) \rangle$, $v \in V, m \in M, \mu \in M'$.

is again a $(V, Y)$-module.

Heisenberg example:
For the choice of conformal vector $\omega_\rho$, we have $F'_\lambda \cong F_{2\rho - \lambda}$

# Motivating tensor products

- In quantum field theory all information is encoded in $n$-point correlation functions.
- In conformal quantum field theory (CFT) these correlation functions are $V$-multilinear functions.
- So we need to understand multilinear algebra for vertex algebras.
- This is also a natural question for commutative algebras.

## Definition: Intertwining operator, $V$-bilinear maps

Let $(V, \Omega, Y, \omega)$ be a vertex operator algebra and $(M_1, Y^{M_1}), (M_2, Y^{M_2}), (M_3, Y^{M_3})$ be $V$-modules. An intertwining operator of type $\binom{M_3}{M_1,\ M_2}$ is a map $\mathcal{Y}_x : M_1 \otimes M_2 \to M_3\{x\}$ such that for all $m_i \in M_i$

- $\mathcal{Y}_x(m_1 \otimes m_2)$ truncates below.
- $\mathcal{Y}_x(L_{-1} m_1 \otimes m_2) = \partial_x \mathcal{Y}_x(m_1 \otimes m_2)$.
- The expansions $Y^{M_3}_z(A \otimes \mathcal{Y}_x(m_1 \otimes m_2)) \sim \mathcal{Y}_x(Y^{M_1}_{z-x}(A \otimes m_1) \otimes m_2) \sim \mathcal{Y}_x(m_1 \otimes Y^{M_2}_z(A \otimes m_2))$ can be identified.

Observations:

- The field map $Y$ is an intertwining operator of type $\binom{V}{V,\ V}$.
- The action $Y^M$ is an intertwining operator of type $\binom{M}{V,\ M}$.
- Intertwining operators are $V$-bilinear maps. All intertwining operators of a given type form a vector space. The field map $Y$ and the action $Y^M$ have a distinguished normalisation due to $Y_z(\Omega \otimes -) = \mathrm{id}$.

# Example: Heisenberg intertwining operators

Recall the Heisenberg Fock spaces $F_\mu$, $\mu \in \mathbb{C}$.
Then $\dim \binom{F_\rho}{F_\mu,\ F_\nu} = \delta_{\rho,\mu+\nu}$ for all $\rho, \mu, \nu \in \mathbb{C}$.
$\binom{F_{\mu+\nu}}{F_\mu,\ F_\nu}$ is spanned by

$$\mathcal{Y}_x^{F_\mu, F_\nu}(p|\mu\rangle \otimes q|\nu\rangle) = x^{\mu\nu} S_\mu \prod_{m\geq 1} \exp\left(\mu \frac{a_{-m}}{m} x^m\right) Y_x^{F_\nu}(p|0\rangle \otimes -) \cdot \prod_{m\geq 1} \exp\left(-\mu \frac{a_m}{m} x^{-m}\right) q|\nu\rangle,$$

where $S_\mu : |\nu\rangle \mapsto |\mu+\nu\rangle$ is the shift operator.

Tensor products pull multilinear algebra back to linear algebra!

## Definition: Fusion product aka vertex algebra tensor product

Let $(V, \Omega, Y, \omega)$ be a vertex operator algebra and $(M_1, Y^{M_1}), (M_2, Y^{M_2})$ be $V$-modules. A fusion product is a triple $(M_1 \boxtimes M_2, Y^{M_1 \boxtimes M_2}, \mathcal{Y}^{M_1,M_2})$, where $(M_1 \boxtimes M_2, Y^{M_1 \boxtimes M_2})$ is a $V$-module and $\mathcal{Y}^{M_1,M_2}$ is an intertwining operator of type $\binom{M_1 \boxtimes M_2}{M_1,\ M_2}$ such that the following universal property holds: For every $V$-module $(X, Y^X)$ and intertwining operator $\mathcal{Y}^X$ of type $\binom{X}{M_1, M_2}$

$$\begin{array}{ccc} M_1 \otimes M_2 & \xrightarrow{\mathcal{Y}^{M_1,M_2}} & M_1 \boxtimes M_2\{z\} \\ & \searrow^{\mathcal{Y}^X} & \Big\downarrow{\scriptstyle \exists! f} \\ & & X\{z\} \end{array}$$

In contrast to linear algebra (or ring theory) constructing $M_1 \boxtimes M_2$ and decomposing into a direct sum of indecomposable modules is extremely hard.

Well chosen categories of modules are tensor categories with respect to $\boxtimes$ with the following structures. [Huang-Lepowsky-Zhang]

- For module homomorphimsms $f: X \to Z$, $g: U \to W$, the morphism $f \boxtimes g$ is uniquely characterised by $(f \boxtimes g) \circ \mathcal{Y}^{X\boxtimes U} = \mathcal{Y}^{Z\boxtimes W} \circ (f \otimes g)$
- $V$ is the tensor identity and the unit isomorphisms are uniquely characterised by $\ell_M \left( \mathcal{Y}_z^{V,M}(a \otimes m) \right) = Y_z^M(a \otimes m)$ and $r_M \left( \mathcal{Y}^{M,V}(m \otimes a) \right) = e^{zL_{-1}} Y_{-z}^M(a \otimes m)$.
- associativity isomorphisms (hardest part!) $A_{M_1,M_2,M_3} \left( \mathcal{Y}_{x_1}^{M_1,M_2\boxtimes M_3}(m_1 \otimes \mathcal{Y}_{x_2}^{M_2,M_3}(m_2 \otimes m_3)) \right) = \mathcal{Y}_{x_2}^{M_1\boxtimes M_2,M_3}(\mathcal{Y}_{x_1-x_2}^{M_1,M_2}(m_1 \otimes m_2) \otimes m_3)$
  All analytic details hidden.
- Braiding isomorphisms uniquely characterised by $c_{M_1,M_2} \left( \mathcal{Y}_x^{M_1,M_2}(m_1 \otimes m_2) \right) = e^{xL_{-1}} \mathcal{Y}_{e^{i\pi}x}^{M_2,M_1}(m_2 \otimes m_1)$

- If the vertex algebra $V$ is conformal (a vertex operator algebra) and the modules are chosen to be compatible with this conformal structure, then there is also a twist $\theta_M = e^{2\pi i L_0}|_M$, which satisfies the balancing equation
  $\theta_{M_1 \boxtimes M_2} = c_{M_1,M_2} \circ c_{M_2,M_1} \circ (\theta_{M_1} \boxtimes \theta_{M_2})$
- Tensor categories of vertex operator algebra modules depend only very weakly on the conformal structure. Only the twist and taking duals depend on the conformal structure.

## Theorem [Huang, Moore-Seiberg]: The Verlinde Conjecture

Let $(V, Y, \Omega, \omega)$ be a vertex operator algebra and $\mathrm{Adm}\, V$ be the category of admissible $V$-modules. If

1. $\dim V_0 = 1, \dim V_{-n} = 0, \dim V_n < \infty,\ n \in \mathbb{N}$,
2. $V$ is simple as a module over itself,
3. $V \cong V^*$, self-dual,
4. $\dim V/c_2(V) < \infty$, (a technical finiteness condition)
5. $\mathrm{Adm}(V)$ is semisimple,

then $\mathrm{Adm}\, V$ is a modular tensor category. Further the action of the modular group on the category (which determines Verlinde's formula) is equal (after a renormalisation) to the action of the modular group on module characters.

# Recap

- Vertex algebras are almost commutative unital algebras with derivations.
- The conformal vector is a choice/structure: there can be 0, 1 or many.
- Vertex algebras admit modules. “Good choices” of module categories admit a tensor (aka fusion) product.
- With the exception of associators, the tensor structure morphisms follow from easy constructions.

# Beyond modular tensor categories

The Verlinde conjecture is about vertex operator algebras that are *maximally nice*. There are many deviations from niceness.

- The category of modules need not be finite (the Heisenberg example is not finite).
- The category of modules need not be semisimple.
- The tensor product need not be left exact (there can be non-flat objects). [Gaberdiel-Runkel-SW]
- The first two deviations above can still be nice in the sense that they admit rigid duals. [Tsuchiya-SW,Allen-SW]
  The third deviation is more fundamentally broken, as non-flat objects cannot have rigid duals.

# Duals of intertwining operators

## Proposition [HLZ]

Let $\mathcal{Y}$ be an intertwining operator of type $\binom{M_3}{M_1,\ M_2}$. Then

$$\begin{aligned} M_1 \otimes M_3' &\to M_2'\{x\} \\ m \otimes \nu &\mapsto \nu(\mathcal{Y}_{x^{-1}}(e^{zL_1}(-z^2)^{L_0} m \otimes -)) \end{aligned}$$

is an intertwining operator of type $\binom{M_2'}{M_1,\ M_3'}$.

Thus $\binom{M_3}{M_1,\ M_2} \cong \binom{M_2'}{M_1,\ M_3'}$.

## Remark

Intertwining operators of type $\binom{M_3}{M_1,\ M_2}$ should be thought of as $\mathrm{Hom}$-spaces of the form $\mathrm{Hom}_V(M_1 \boxtimes M_2, M_3)$.

## Defininition: Grothendieck Verdier categories

Let $\mathcal{C}$ be a monoidal (abelian linear) category.

1. An object $K \in \mathcal{C}$ is called *dualising*, if for all $Y \in \mathcal{C}$ the functor $\mathrm{Hom}_V(- \otimes Y, K)$ is representable, that is,

$$\mathrm{Hom}_V(X \otimes Y, K) \cong \mathrm{Hom}_V(X, GY),$$

   and if the so defined contravariant functor $G : \mathcal{C} \to \mathcal{C}$ is an anti-equivalence.
2. A monoidal category $\mathcal{C}$ together with a choice of dualising object $K \in \mathcal{C}$ is called a *Grothendieck-Verdier* or $*$*-autonomous* category.

## Theorem [Allen-Lentner-Schweigert-SW]

Let $(V, Y)$ be a vertex operator algebra and let Rep$(V)$ be choice of modules to which the HLZ tensor product theory applies which is in addition closed under taking duals. Then $V'$ is a dualising object for Rep$(V)$ and (Rep$(V), V')$ is a Grothendieck-Verdier category (actually ribbon Grothendieck-Verdier).

# Why am I excited about this?

Grothendieck-Verdier categories have many appealing features

- The appear to be describe the *natural* duality structure of vertex operator algebra modules and the can accommodate all non-nice features mentioned previously.
- They admit two tensor products $X \otimes Y$ and $X \bullet Y = G(G^{-1}Y \otimes G^{-1}X)$, where $\otimes$ is right exact and $\bullet$ is left exact. In particular there are distributor morphisms

  $$\partial^l_{X,Y,Z} : X \otimes (Y \bullet Z) \to (X \otimes Y) \bullet Z, \qquad \partial^r_{X,Y,Z} : (X \bullet Y) \otimes Z \to X \bullet (Y \otimes Z),$$

  which need not be isomorphisms. [Fuchs-Schaumann-Schweigert-SW]
- $\otimes$ admits inner Homs (right adjoints) and $\bullet$ admits inner coHoms (left adjoints which allow the construction of algebras and coalgebras (even Frobenius algebras) in $\mathcal{C}$ (crucial for conformal field theory).

Thank you!

# For further reading I


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# For further reading II


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