# INVOLUTIVE SOLUTIONS OF THE PENTAGON EQUATION

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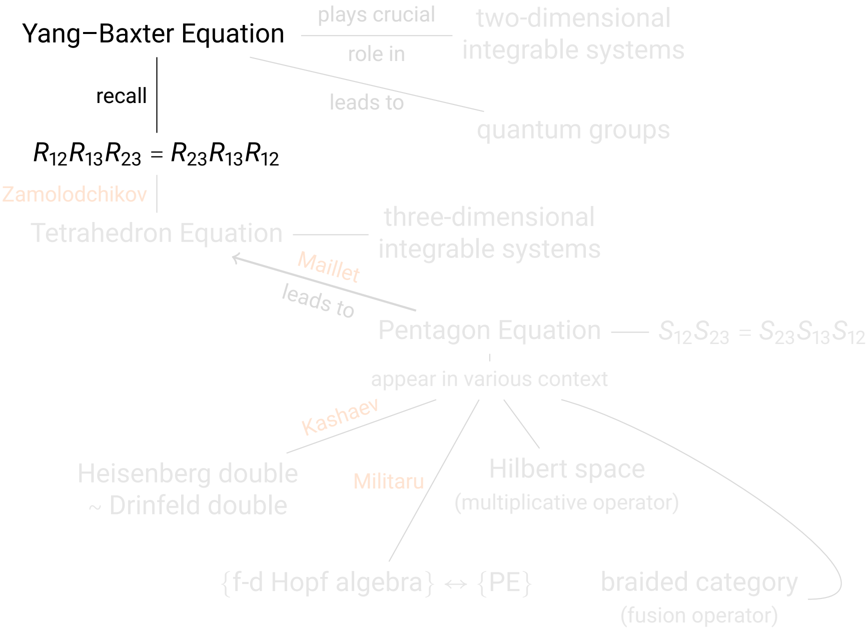
VRIJE UNIVERSITEIT BRUSSEL

# OVERVIEW

1. The Pentagon Equation
2. Set-theoretic solutions of the Pentagon Equation
3. Some general results on bijective solutions of PE
4. Involutive solutions of the PE
5. The retraction of inovlutive solutions of PE
6. The extensions of involutive solution of the PE

# MOTIVATION


Yang–Baxter Equation
plays crucial role in
two-dimensional integrable systems
leads to
quantum groups
recall
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appear in various context
Kashaev
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~ Drinfeld double
Militaru
Hilbert space
(multiplicative operator)
{f-d Hopf algebra} ↔ {PE}
braided category
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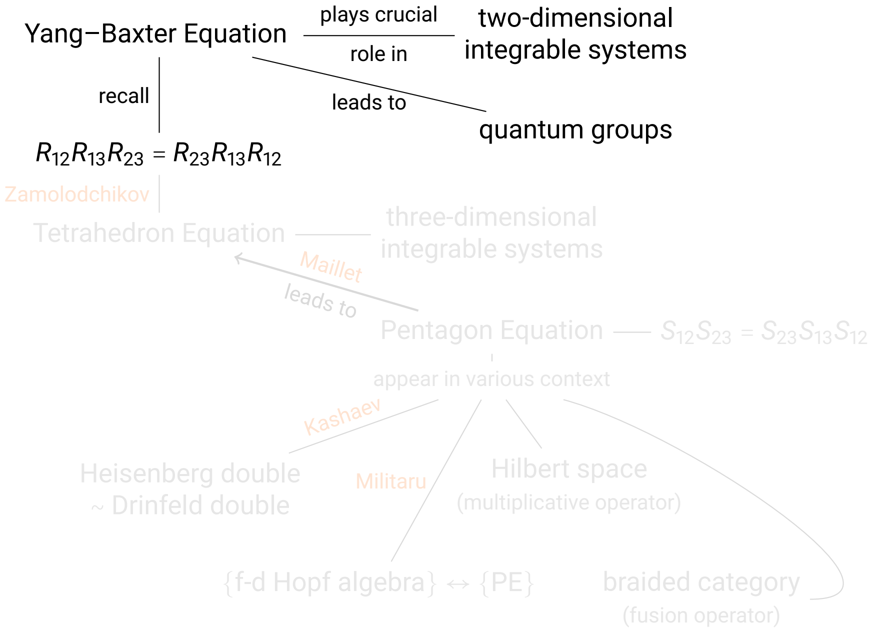
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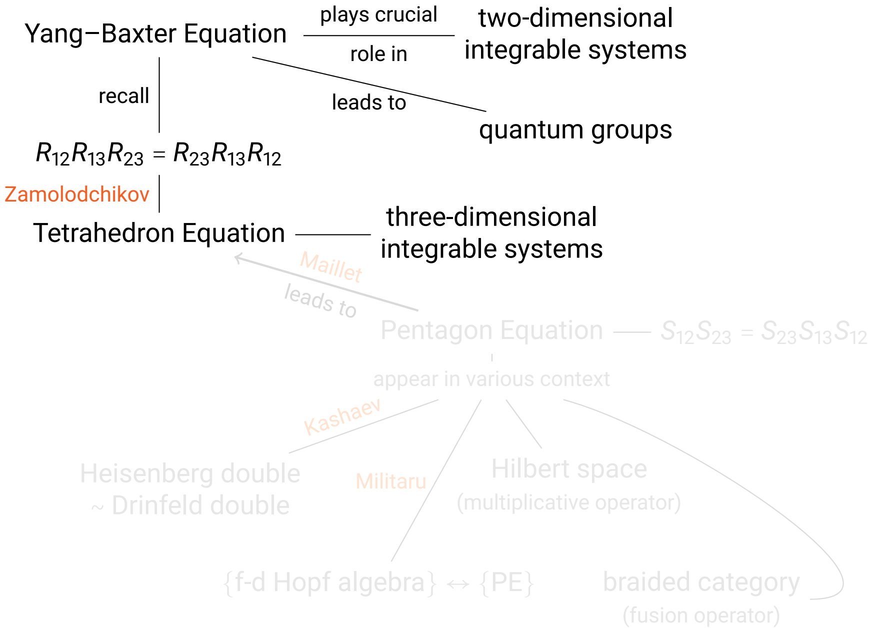
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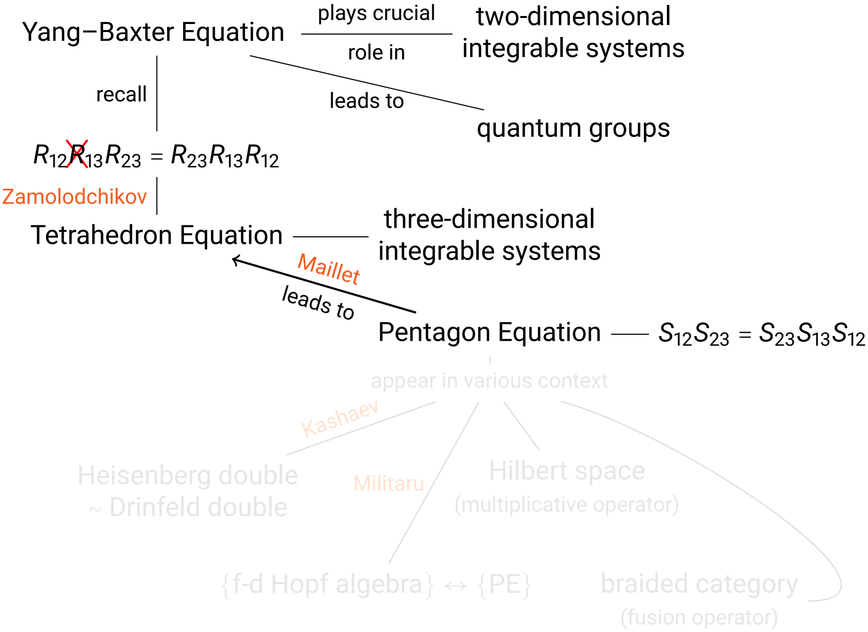
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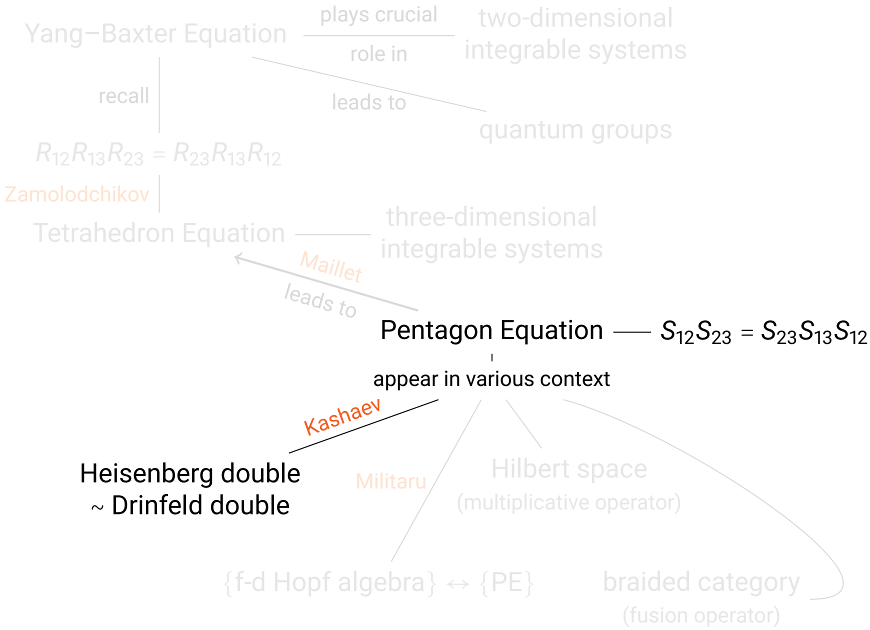
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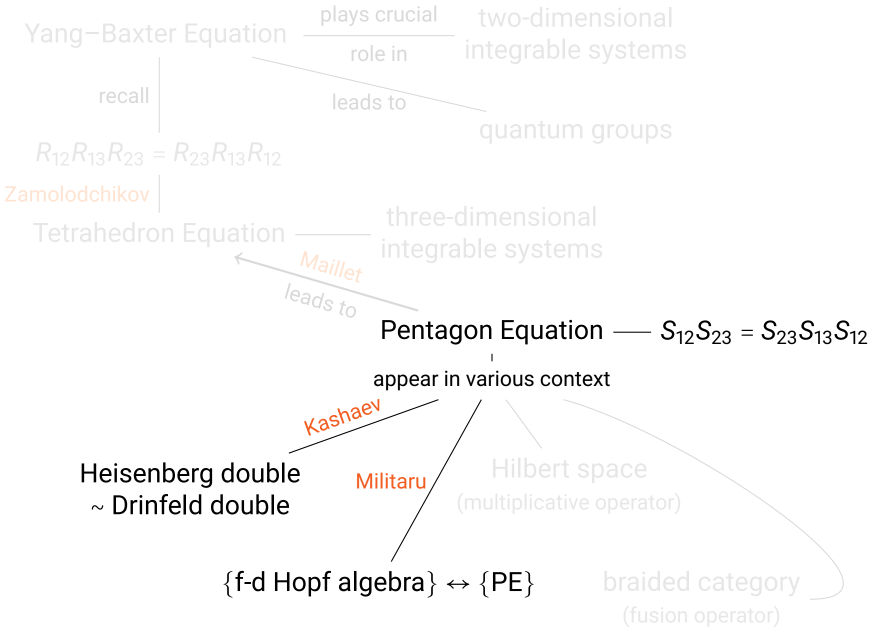
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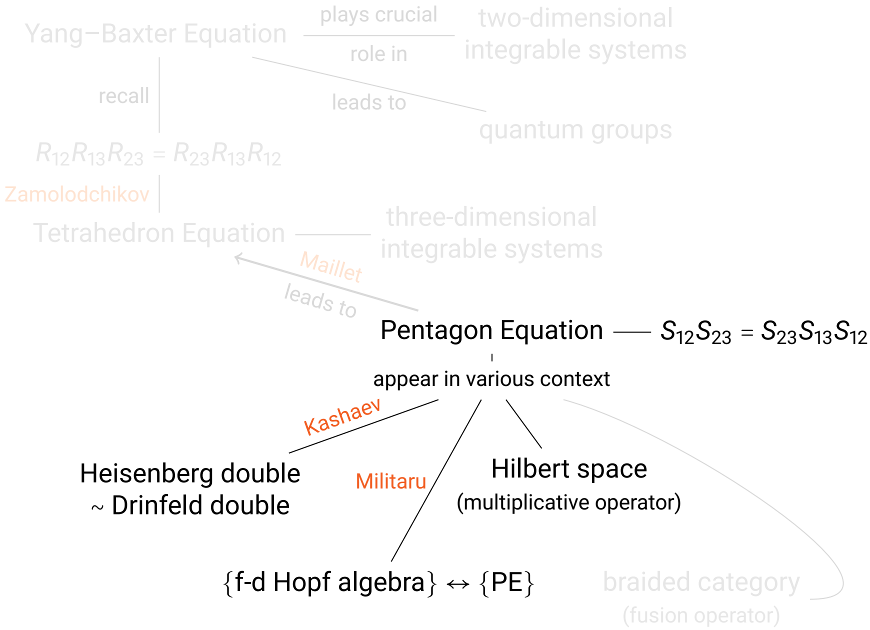
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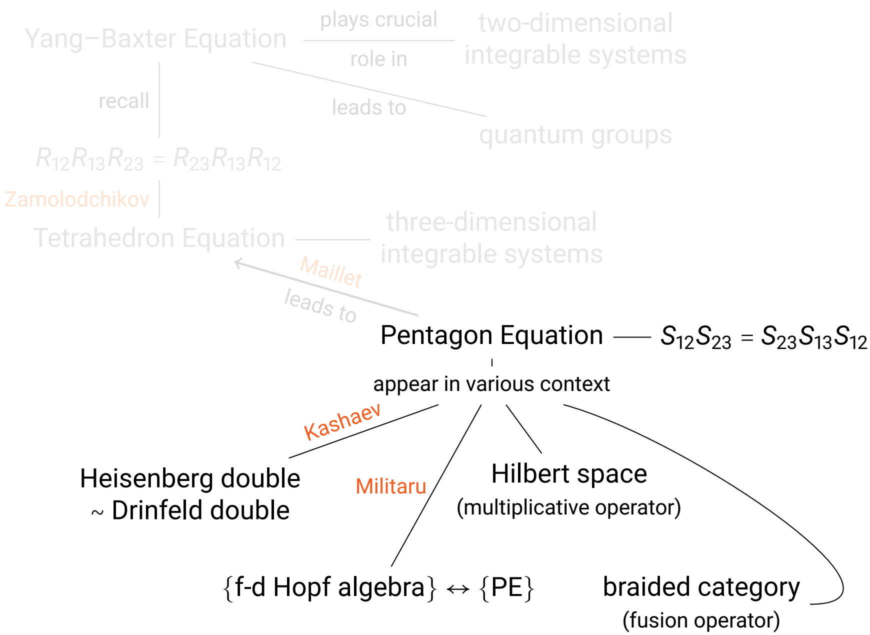
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Set-theoretic solutions of PE received a large interest

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## SET-THEORETIC SOLUTIONS OF THE PE

### DEFINITION

$S$ a set $\qquad s: S \times S \to S \times S$

$(S, s)$ is a set-theoretic solution of the Pentagon Equation if

$$s_{23}s_{13}s_{12} = s_{12}s_{23}$$

$$s_{12} = s \times \mathrm{id} \qquad s_{23} = \mathrm{id} \times s \qquad s_{13} = (\tau \times \mathrm{id})(\mathrm{id} \times s)(\tau \times \mathrm{id})$$

- $(S, s)$ is finite if $S$ is a finite set
- $(S, s)$ is bijective if $s$ is a bijective map
- $(S, s)$ bijective of finite order if $\exists n > 0$ such that $s^n = \mathrm{id}$
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## A CHARACTERIZATION

Write $s(x,y) = (x \cdot y, \theta_x(y))$ then

$(S,s)$ is a solution of the PE $\iff$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$
$$\theta_x(y) \cdot \theta_{x \cdot y}(z) = \theta_x(y \cdot z)$$
$$\theta_{\theta_x(y)} \theta_{x \cdot y} = \theta_y$$

Hence, $(S, \cdot)$ must be a semigroup.
We denote the multiplication in $S$ as a concatenation, i.e., $x \cdot y = xy$.

$(S,s)$ is a solution of the PE $\iff$ $t = \tau s \tau : S \times S \to S \times S$ satisfies $\underline{t_{12} t_{13} t_{23} = t_{23} t_{12}}$

$t$ is called a solution of the Reversed Pentagon Equation (RPE).

$(S,s)$ is a bijective solution of the PE $\iff$ $s^{-1}$ satisfies the RPE

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# SET-THEORETIC SOLUTIONS OF THE PE

## EXAMPLES

- $S$ a semigroup
- $f \in \mathrm{End}(S)$, $f^2 = f$

$s(x,y) = (xy, f(y))$ is a solution of the PE.

- $S$ a set
- $f, g \in \mathrm{Map}(S,S)$

$f^2 = f, g^2 = g, fg = gf \implies s(x,y) = (f(x), g(y))$ solution of PE and RPE

- $G$ group with $\exp(G) < \infty$
- $E = \{1, \ldots, n\}$
- $\sigma \in \mathrm{Sym}(n)$ s.t $\forall i \in E$ $\sigma^{\sigma(i)+1} = \sigma^i$
- $S = E \times G$

$$s((i,a),(j,b)) = ((i,ab),(\sigma^i(j),b))$$

is a bijective solution of the PE s.t. $\mathrm{order}(s) = \mathrm{lcm}(\mathrm{order}(\sigma), \exp(G))$

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## SOME GENERAL RESULTS ON BIJECTIVE SOLUTIONS

- $(S, s)$ bijective solution of the PE

$$\forall x, y \in S,\ \exists u, v \in S \ \text{s.t.} \quad \underline{s(u,v) = (x,y)}$$

$$uv = x \implies S = S^2$$

$$y = \theta_u(v) \implies \theta_y\theta_x = \theta_{\theta_u(v)}\theta_{uv} = \theta_v \in T \implies T \text{ is a semigroup}$$

- $(S, s)$ bijective solution of the PE of finite order

$\exists n > 0$ s.t. $s^n = \mathrm{id} \implies \forall x, y \in S\ \exists z \in S$ s.t. $xyz = x \implies \forall x, y\colon x \in SyS$.
Hence, $S$ is a simple semigroup.

## SOME GENERAL RESULTS ON BIJECTIVE SOLUTIONS

- $(S, s)$ bijective solution of the PE

$$\forall x, y \in S, \; \exists u, v \in S \text{ s.t. } \underline{s(u, v) = (x, y)}$$

$uv = x$ ⟵ (from $s(u,v) = (x,y)$)

$\downarrow$

$S = S^2$

From $s(u,v) = (x,y)$:

$\downarrow$

$y = \theta_u(v)$

$\downarrow$

$\theta_y\theta_x = \theta_{\theta_u(v)}\theta_{uv} = \theta_v \in T$

$\downarrow$

$T$ is a semigroup

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- $(S, s)$ bijective solution of the PE

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$$uv = x$$
$$\downarrow$$
$$S = S^2$$

$$\downarrow$$
$$y = \theta_u(v)$$
$$\downarrow$$
$$\theta_y \theta_x = \theta_{\theta_u(v)} \theta_{uv} = \theta_v \in T$$
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## LEFT GROUPS

- $E$ is a left zero semigroup
- $G$ a group

$E \times G$ is called left group

$$(i, g)(j, h) = (i, gh), \forall i, j \in E, g, h \in G$$

Every left group is left simple, and is right cancellative.

**Remark**

A semigroup $E$ is called a left zero semigroup if $xy = x$ for all $x, y \in E$.

- $(S, s)$ a solution of the PE and the RPE ← E.g. when $(S, s)$ is involutive.

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# SOLUTIONS OF PE OVER A GROUP

- $G$ a group
- $s(x, y) = (xy, y)$ is a bijective solution of the PE.

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The order of $s$ is $\exp(G)$.

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# INVOLUTIVE SOLUTIONS OF THE PE

- $E$ a left zero semigroup
- $(E, s_E)$ an involutive solution of the PE on $E$

$$s_E(i,j) = (i, \theta_i(j))$$

- $G$ an elementary abelian 2-group
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$$s_G(x,y) = (xy, y))$$

- $S := G \times E$

the map $s : S \times S \to S \times S$ defined by

$$s((i,x),(j,y)) = ((i,xy),(\theta_i(j),y))$$

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## AS DIRECT PRODUCT OF SOLUTIONS

- $(S, s)$ an involutive solution of the PE
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- $\exists (E, s_E)$ an involutive solution of the PE on $E$

s.t.

$$(S, s) = (E, s_E) \times (G, s_G),$$

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# INVOLUTIVE SOLUTIONS OF THE PE

## A REDUCTION

Hence, each involutive solution of the PE $(S, s)$ is composed of solutions $s_E$ and $s_G$ on a left zero semigroup $E$ and on an elementary abelian 2-group $G$.

The solution $s_G$ is unique.

$\downarrow$

the description of all involutive set-theoretic solutions $(S, s)$ of the PE on a semigroup $S$ can be reduced to the description of solutions on a left zero semigroup.

# THE RETRACTION OF INVOLUTIVE SOLUTIONS OF PE

## DEFINITION

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Define the equivalence relation $\sim$ on $S$ called retraction

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$$\theta_{xy} = \theta_x \iff xy \sim x$$

$$x_1 \sim y_1 \text{ and } x_2 \sim y_2 \implies x_1x_2 \sim x_1 \sim y_1 \sim y_1y_2 \implies \sim \text{ is a congruence of } S$$

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$$\theta_{\theta_x(y)} = \theta_x \theta_y$$

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This allows us to define a map $\overline{s}\colon \overline{S} \times \overline{S} \to \overline{S} \times \overline{S}$

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Hence, $\mathrm{Ret}(S, s) = (\overline{S}, \overline{s})$ is a solution of PE called retract

# IRRETRACTABLE INVOLUTIVE SOLUTIONS OF THE PE

## THE RETRACT IS IRRETRACTABLE

- $(S, s)$ an involutive solution of PE.

$(S, s)$ is irretractable if $(S, s) = \mathrm{Ret}(S, s)$.

$$\overline{\theta}_{\overline{x}} = \overline{\theta}_{\overline{y}} \iff \overline{\theta_x(z)} = \overline{\theta_y(z)} \text{ for all } z \in S$$
$$\iff \theta_{\theta_x(z)} = \theta_{\theta_y(z)} \text{ for all } z \in S$$
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# IRRETRACTABLE INVOLUTIVE SOLUTIONS OF THE PE

## THE IRRETRACTABLE SOLTION IS "UNIQUE"

- $(A, +)$ an elementary abelian 2-group.
- Define $t\colon A \times A \to A \times A$ by $t(x, y) = (x, x + y)$.

Then $(A, t)$ is an irretractable involutive solution of the PE.

- $(S, s)$ an irretractable involutive solution of PE
- $\exists +$ s.t. $(S, +)$ is an elementary abelian 2-group
- and $s(x, y) = (x, x + y)$

The solution $(A, t)$ on an elementary abelian 2-group $(A, +)$ will be denoted as $(A, t_A)$.

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# INVOLUTIVE SOLUTIONS OF PE

## THE EXTENSIONS

- $(A, +)$ an elementary abelian 2-group
- $(A, t_A)$ the irretractable involutive solution of the PE on $A$
- $X$ a non-empty set
- $\sigma: A \to \mathrm{Sym}(X)$
- $S = X \times A$

Define on $S \times S$

$$s((x,a),(y,b)) = ((x,a),(\sigma_{a+b}\sigma_b^{-1}(y), a+b)).$$

- $(S, s)$ is an involutive solution of the PE.
- $\mathrm{Ret}(S, s) = (A, t_A)$.

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## SOLUTIONS AS EXTENSIONS

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# INVOLUTIVE SOLUTIONS OF PE

## A DESCRIPTION

- $(S, s)$ an involutive solution of the PE.
- $\exists A, G$ elementary abelian 2-groups
- $X$ a non-empty set
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s.t. $S = X \times A \times G$ and

$$(S, s) = \mathrm{Ext}_X^{\sigma}(A, t_A) \times (G, s_G)$$

- $(A, t_A)$ is the unique irretractable invulutive solution of PE on $A$
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# INVOLUTIVE SOLUTIONS OF PE

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When are two extensions isomorphic as solutions?

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