

Bijective set-theoretic solutions of the Pentagon Equation

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Al@Bicocca Take Away 12 November 2021



Overview

The Pentagon Equation (PE)

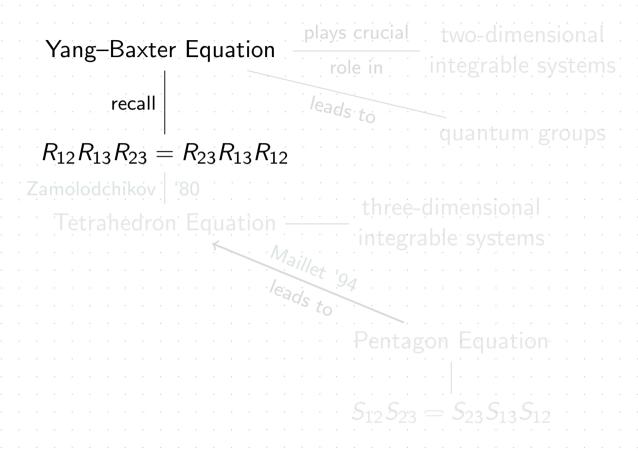
Set-theoretic solutions of the PE

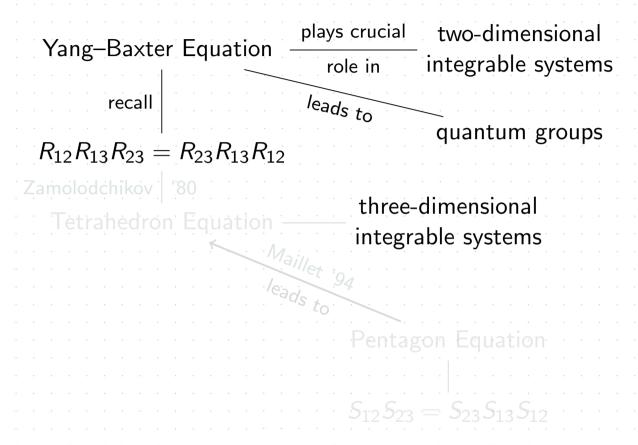
Some general results on bijective solutions of the PE

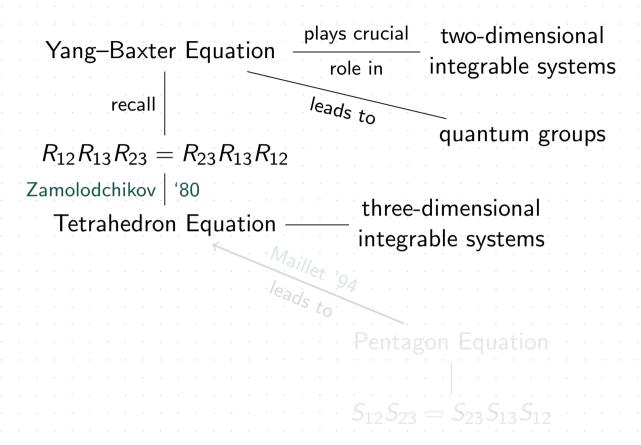
Involutive solutions of the PE

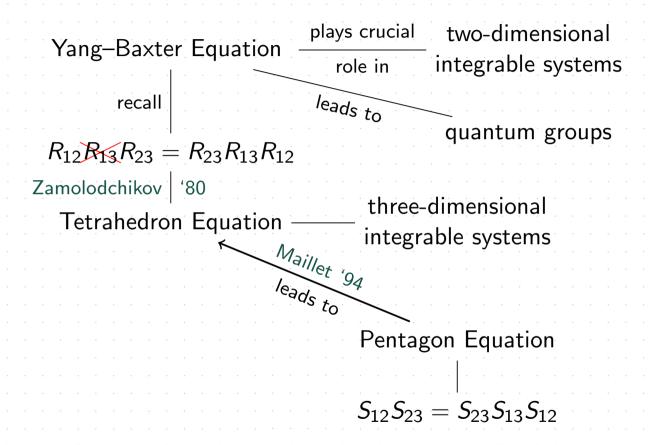
The retraction of inovlutive solutions of the PE

The extensions of involutive solution of the PE

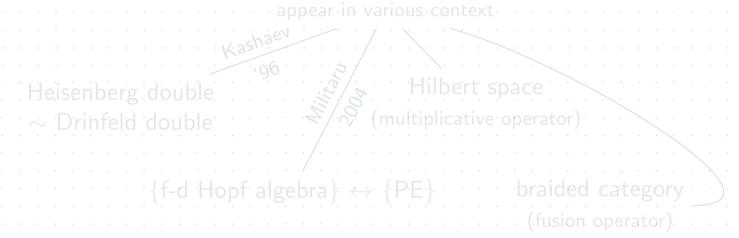






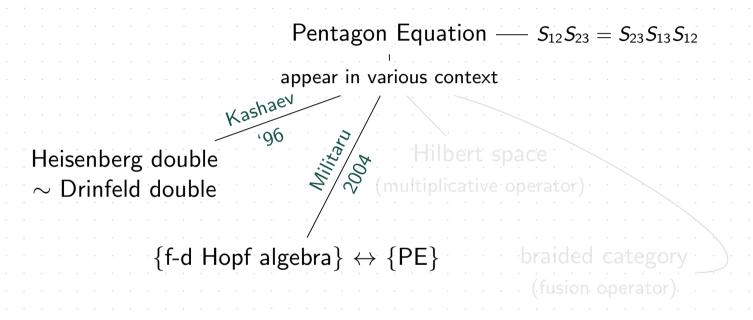


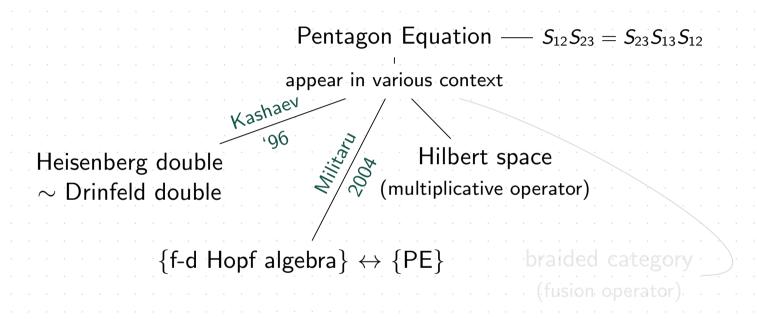
Pentagon Equation — $S_{12}S_{23} = S_{23}S_{13}S_{12}$

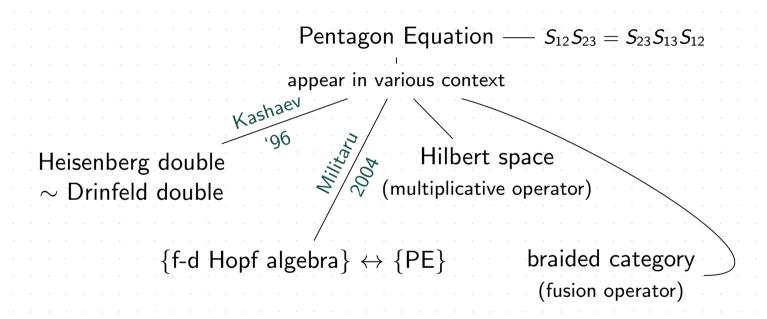


Pentagon Equation — $S_{12}S_{23} = S_{23}S_{13}S_{12}$ appear in various context

Heisenberg double — Hilbert space — Drinfeld double — (multiplicative operator) $\{f-d \text{ Hopf algebra}\} \leftrightarrow \{PE\}$ braided category — (fusion operator)







The problem

► Study set-theoretic version of the YBE and the PE

[Drinfeld, 1992]

► Set-theoretic solutions of PE received a large interest

[Baaj Skandalis, 2003] [Jiang Liu, 2005]

[Kashaev2011] [Kashaev Reshetikhin, 2007]

► The study of set-theoretic solutions of PE form a pure algebraic viewpoint

[Catino, Mazzotta, Miccoli, 2020]

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The problem and our aim

Problem

Description of set-theoretic solutions of the Pentagon Equation

Aim

Description of all involutive set-theoretic solutions of the Pentagon Equation



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Set-theoretic solutions of the PE

Definition

Let S be 2 set and s: Sxs -> 5xs be 2 map.

(S,0) is a (set-terrette) solution of the pentagon

EQUATION if in
$$S \times S \times S$$
 the following is satisfied

 $\Delta_{23} \Delta_{13} \Delta_{12} = \Delta_{12} \Delta_{23}$
 $\Delta_{23} = id \times \Delta$, $\Delta_{12} = \delta_{12} \Delta_{23}$
 $\Delta_{3} = id \times \Delta$, $\Delta_{12} = \delta_{12} \Delta_{23}$

- · Finite if S is finite
- · Byective if s is byective
- · Bijective of finite order if Frell on = id
- · Involutive if size id



Set-theoretic solutions of the PE

A characterization

Set-theoretic solutions of the PE

and solutions of the RPE

PE 323/38512 = 512/523 12/3/23 = 523/13/12

$$(S,s)$$
 bijective solution $= (S,S')$ is a solution of the PE of the RPE

bj = so



Set-theoretic solutions of the PE: Examples

• 5 set
$$f(g: Mop(s,s))$$

 $s(x,y) = (f(x), g(y)) = g^2 = g + f^2 = f$
is a solution $g^2 = fg$

· (S,.) semigroup

$$S(x,y)=(xy, b(y)) = b \in End(s, \cdot)$$

is a solution of the $b^2=b$

Set-theoretic solutions of the PE: Examples

The bijective solutions of PE over a group

• G Scoup $S_{a}(x,y) = (xy,y)$ (G, &) is the unique by ective solution

Some general results on bijective solutions

(S,5) = bijective solution of the PE

$$\forall x, y \in S \Rightarrow \exists |y| v \in S \quad (x,y) = b(y,v)$$

(uv, $\partial y(v)$)

- $2 = 40 = 8^2 =$
- · Oyon = Oyun dun = On T= {Ox: x ∈ S} is a semigroup

(S, s) is of Finite orde Jn s.t. sh=id

$$\forall x, y \in S \quad \exists z \quad xyz = z$$

 $SyS = S \quad \Rightarrow S \quad is \quad a simple semigroup$

Some general results on bijective solutions of PE

Left groups

$$E - left$$
 zero semigeoup $\forall x,y xy = x$
 $G - g z o u p$
 $S = E \times G$ is LEFT GROUP
 $(i,g)(j,g) = (i,gg)$

- (S,5) is a bijective solution of the PE and RPE (E.g. 8=id)
 Then S is a left group
- S is a left zero semigraup and a is of finite order then $\theta_{\chi} \in Sym(S)$ and $\theta_{\chi} = id$ or tpf



Involutive solutions of the PE

A retractable involutive solution of the CDE

Let
$$E$$
 be left zero semigroup

 G be a grementary abelian bargin)= (g_n, h)
 (E, bE) involutive solution on E $b(i,j)=(i,b(i))$
 $S=E\times G$
 $b((i,g),(j,h))=((i,g_n),(\theta_i(j),h))$

Then (S,b) is a solution of the PE called the $BIRECT$ PRODUCT of (E,bE) and (G,bG)

Involutive solutions of the PE

As direct product of solutions

(S,A) involutive solution of the DE
$$\exists E \mid \text{left}$$
 zero semigroup $\exists G$ group $\exists G$ group $S.+.$ $S = E \times G$ $\Rightarrow S \mid E \times E$ $\Rightarrow S \mid E \mid E \mid E$ $\Rightarrow S \mid E$ $\Rightarrow S \mid E \mid E$ $\Rightarrow S \mid E$ $\Rightarrow S \mid E \mid E$ $\Rightarrow S \mid E \mid E$ $\Rightarrow S \mid E$ $\Rightarrow S \mid E$ $\Rightarrow S \mid E \mid E$ $\Rightarrow S \mid E$

Involutive solutions of the PE

A reduction

Hence, each involutive solution of the PE (S, s) is composed of solutions s_E and s_G on a left zero semigroup E and on an elementary abelian 2-group G.

The solution s_G is unique.

the description of all involutive set-theoretic solutions (S, s) of the PE on a semigroup S can be reduced to the description of solutions on a left zero semigroup.

 \triangleright (S, s) an involutive solution of the PE.

Define the equivalence relation \sim on S called retraction

$$\theta_{x} \sim y \iff \theta_{x} = \theta_{y}.$$

From the description of involutive solutions we get

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Define the equivalence relation \sim on S called retraction

$$f_X \sim y \iff \theta_X = \theta_y$$
.

From the description of involutive solutions we get

$$heta_{xy} = heta_x \iff xy \sim x$$
 $x_1 \sim y_1 \text{ and } x_2 \sim y_2 \implies x_1x_2 \sim x_1 \sim y_1 \sim y_1y_2$
 $\implies \sim \text{ is a congruence of } S$

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$$y \times y \sim y \iff \theta_x = \theta_y$$
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$$heta_{ heta_{ imes}(y)} = heta_{ imes} heta_{y}$$
 $imes_{x_{1}} \sim y_{1} ext{ and } x_{2} \sim y_{2} \implies heta_{ heta_{x_{1}}(y_{1})} = heta_{x_{1}} heta_{y_{1}} = heta_{x_{2}} heta_{y_{2}} = heta_{ heta_{x_{2}}(y_{2})}$ $\implies heta_{x_{1}}(y_{1}) \sim heta_{x_{2}}(y_{2})$

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$$f_X \sim y \iff \theta_X = \theta_y$$
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- ightharpoonup \sim is a congruence of S
- $\blacktriangleright \theta_{x_1}(y_1) \sim \theta_{x_2}(y_2)$

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Define the equivalence relation \sim on S called retraction

$$f(x) \sim y \iff \theta_x f = \theta_y f.$$

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- ightharpoonup \sim is a congruence of S
- $\blacktriangleright \theta_{x_1}(y_1) \sim \theta_{x_2}(y_2)$

This allows us to define a map $\overline{s} \colon \overline{S} \times \overline{S} \to \overline{S} \times \overline{S}$

$$\overline{s}(\overline{x},\overline{y})=(\overline{x},\overline{\theta}_{\overline{x}}(\overline{y})).$$

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Hence, $Ret(S, s) = (\overline{S}, \overline{s})$ is a solution of PE called retract



Irretractable involutive solutions of the PE

The retract is irretractable

 \triangleright (S, s) an involutive solution of PE.

$$(S,s)$$
 is irretractable if $(S,s) = Ret(S,s)$.

$$\overline{\theta}_{\overline{x}} = \overline{\theta}_{\overline{y}} \iff \overline{\theta_{x}(z)} = \overline{\theta_{y}(z)} \text{ for all } z \in S$$

$$\iff \theta_{\theta_{x}(z)} = \theta_{\theta_{y}(z)} \text{ for all } z \in S$$

$$\iff \theta_{x}\theta_{z} = \theta_{y}\theta_{z} \text{ for all } z \in S$$

$$\iff \overline{\theta}_{x} = \overline{\theta}_{y}$$

$$\iff \overline{x} = \overline{y}.$$

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ightharpoonup Ret(S,s) is an irretractable involutive solution of the PE.

Irretractable involutive solutions of the PE

The irretractable soltion is "unique"

(A,t) elementary abelian 2-group

$$t_{A}(x,y) = (x_{1}x+y)$$
 (*)

is a irretractable solution of the PE

involutive

 $t_{A} = x_{1}x+y$
 $t_{A} = x_{2}x=x$

If
$$(S,4)$$
 is an irretractable solution then $J+$ $(S,+)$ is an elementary abelian 2-group $(S,\delta)^2(S,\xi_S)$

The extensions

- \triangleright (A, +) an elementary abelian 2-group
- \triangleright (A, t_A) the irretractable involutive solution of the PE on A
- \triangleright X a non-empty set
- \triangleright $S = X \times A$

Define on $S \times S$

$$s((x,a),(y,b)) = ((x,a),(\sigma_{a+b}\sigma_b^{-1}(y),a+b))$$

- \triangleright (S, s) is an involutive solution of the PE
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Solutions as extensions

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- $ightharpoonup \exists \sigma : A \rightarrow \operatorname{Sym}(X)$

s.t
$$S = X imes A$$
 and $(S,s) = \operatorname{\mathsf{Ext}}_X^\sigma(A,t_A)$

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A description

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- $ightharpoonup \exists A, G \text{ elementary abelian 2-groups}$
- ► X a non-empty set
- ▶ $\sigma: A \rightarrow \operatorname{Sym}(X)$ a map

s.t.
$$S = X \times A \times G$$
 and

$$(S,s) = \operatorname{Ext}_X^{\sigma}(A,t_A) \times (G,s_G)$$

- \triangleright (A, t_A) is the unique irretractable invulutive solution of PE on A
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Moreover,
$$Ret(S, s) = (A, t_A)$$



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Isomorphic extensions

When are two extensions isomorphic as solutions?

- \triangleright (S,s) and (S',s') solutions of PE
- ightharpoonup f: S
 ightharpoonup S' a map

f is an isomorphism if f is bijective and $(f \times f)s = s'(f \times f)$

- \triangleright (A, +) an elementary abelian 2-group
- X a non-empty set
- $ightharpoonup \sigma: A
 ightharpoonup \operatorname{Sym}(X)$ and $\rho: A
 ightharpoonup \operatorname{Sym}(X)$ maps

Then $\operatorname{Ext}_X^{\sigma}(A,t_A)$ and $\operatorname{Ext}_X^{\rho}(A,t_A)$ are isomorphic

$$\{\text{involutive solutions}\} \xrightarrow{\text{bijective}} \left\{ X \times A \times G \mid \begin{array}{c} X \neq \emptyset \\ A, G \text{ elem. ab. 2-group} \end{array} \right.$$



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$$\{\text{involutive solutions}\} \stackrel{\text{bijective}}{\longleftrightarrow} \left\{ X \times A \times G \mid \begin{array}{c} X \neq \emptyset \\ A, G \text{ elem. ab. 2-group} \end{array} \right\}$$

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Thanks for your attention!