Bijective set-theoretic solutions of the Pentagon Equation

Ilaria Colazzo
I.Colazzo@exeter.ac.uk

AI@Bicocca Take Away
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Overview

The Pentagon Equation (PE)

Set-theoretic solutions of the PE

Some general results on bijective solutions of the PE

Involutive solutions of the PE

The retraction of inovlutive solutions of the PE

The extensions of involutive solution of the PE

Motivation (I)

Yang–Baxter Equation $\xrightarrow[\text{role in}]{\text{plays crucial}}$ two-dimensional integrable systems

leads to quantum groups

recall

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Zamolodchikov | '80

Tetrahedron Equation — three-dimensional integrable systems

Maillet '94

leads to

Pentagon Equation

$$S_{12}S_{23} = S_{23}S_{13}S_{12}$$

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Pentagon Equation — $S_{12}S_{23} = S_{23}S_{13}S_{12}$

appear in various context

Kashaev '96

Heisenberg double
$\sim$ Drinfeld double

Militaru 2004

{f-d Hopf algebra} $\leftrightarrow$ {PE}

Hilbert space
(multiplicative operator)

braided category
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- Study set-theoretic version of the YBE and the PE

  [Drinfeld, 1992]

- Set-theoretic solutions of PE received a large interest

  [Baaj Skandalis, 2003] [Jiang Liu, 2005]

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- The study of set-theoretic solutions of PE form a pure algebraic viewpoint

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The problem and our aim

Problem

Description of set-theoretic solutions of the Pentagon Equation

Aim

Description of all involutive set-theoretic solutions of the Pentagon Equation

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Set-theoretic solutions of the PE

Definition

Let S be a set and $s: S\times S \to S\times S$ be a map.
(S,s) is a (SET-TEORETIC) SOLUTION OF THE PENTAGON EQUATION if in $S\times S\times S$ the following is satisfied

$$\boxed{s_{23}\, s_{13}\, s_{12} = s_{12}\, s_{23}}$$

$s_{23} = id \times s$, $\quad s_{12} = s \times id \quad$ $s_{13} = (\tau \times id)(id \times s)(\tau \times id)$

(S,s) is

- Finite if S is finite
- Bijective if s is bijective
- Bijective of finite order if $\exists n \in \mathbb{N}$ $s^n = id$
- Involutive if $s^2 = id$

Set-theoretic solutions of the PE

A characterization

S set $\circ$ $s : S \times S \to SS$. Write

$$s(x, y) = (x \cdot y, \theta_x(y))$$

$\theta_x : S \to S$

- $\cdot$ binary operation

s is a solution of PE $\iff$ $(S, \cdot)$ is a semigroup

$$\theta_x(yz) = \theta_x(y)\theta_{xy}(z) \leftarrow$$

$$\theta_{\theta_x(y)}\theta_{xy} = \theta_y$$

Set-theoretic solutions of the PE

and solutions of the RPE

PE

$s_{23} s_{13} s_{12} = s_{12} s_{23}$

$t = \tau r \tau$ is a solution of

$t_{12} t_{13} t_{23} = t_{23} t_{12}$

Reversed pentagon equation (RPE)

YBE

$r_{12} r_{13} r_{23} = r_{23} r_{13} r_{12}$

$\tau r \tau$

$\tau(x, y) = (y, x)$

(S, s) bijective solution of the PE $\Longleftrightarrow$ (S, s^{-1}) is a solution of the RPE

$s^2 = id$

Set-theoretic solutions of the PE: Examples

- S set $\quad f, g : \mathrm{Map}(S, S)$

  $s(x, y) = (f(x), g(y))$ is a solution $\iff g^2 = g \quad f^2 = f$

  $gf = fg$

- $(S, \cdot)$ semigroup

  $s(x, y) = (xy, \vartheta(y))$ is a solution of PE $\iff \vartheta \in \mathrm{End}(S, \cdot)$

  $\vartheta^2 = \vartheta$

Set-theoretic solutions of the PE: Examples

The bijective solutions of PE over a group

- G group

$$s_G(x, y) = (xy, y)$$

(G, s_G) is the unique bijective solution

Some general results on bijective solutions

(S,s) a bijective solution of the PE

$$\forall x,y \in S \Rightarrow \exists!\, u,v \in S \quad (x,y) = s(u,v) = (uv, \sigma_u(v))$$

- $x = uv \Rightarrow S^2 = S$
- $\sigma_y \sigma_x = \sigma_{\sigma_u(v)} \sigma_{uv} = \sigma_v$

  $T = \{\sigma_x : x \in S\}$ is a semigroup

(S,s) is of finite orde $\exists n$ s.t. $s^n = id$

$\forall x,y \in S \quad \exists z \quad xyz = x$

$SyS = S \Rightarrow S$ is a simple semigroup

Some general results on bijective solutions of PE

Left groups

E - left zero semigroup $\quad \forall x,y \quad xy = x$

G - group

$S = E \times G$ is LEFT GROUP

$$(i,g)(j,h) = (i,gh)$$

- (S,s) is a bijective solution of the PE and RPE ($\in$.g. $s^2 = id$)
  Then S is a left group

- S is a left zero semigroup and s is of finite order then $\theta_x \in Sym(S)$ and $\theta_x = id$ or fpf

Involutive solutions of the PE

~~A retractable involutive solution of the PE~~

Let E be left zero semigroup

G be a elementary abelian 2-group $s_G(g,h)=(gh,h)$

(E,s_E) involutive solution on E $\quad s(i,j)=(i,\vartheta_i(j))$

$S=E\times G$

$$s((i,g),(j,h))=((i,gh),(\vartheta_i(j),h))$$

Then (S,s) is a solution of the PE called the DIRECT PRODUCT of (E,s_E) and (G,s_G)

Involutive solutions of the PE

As direct product of solutions

(S, s) involutive solution of the PE

$\exists E$ left zero semigroup
$\exists G$ group

s.t. $S = E \times G$

$$s_E = \underset{\uparrow}{} s_{S|_{E\times E}}^{\#}$$

$$S = (E, s_E) \times (G, s_G)$$

$T = \{\theta_x : x \in S\}$ is an elementary abelian 2-group

Involutive solutions of the PE

A reduction

Hence, each involutive solution of the PE (S, s) is composed of solutions s_E and s_G on a left zero semigroup E and on an elementary abelian 2-group G.

The solution s_G is unique.

$$\downarrow$$

the description of all involutive set-theoretic solutions (S, s) of the PE on a semigroup S can be reduced to the description of solutions on a left zero semigroup.

The retraction of involutive solutions of PE

Definition

- (S, s) an involutive solution of the PE.

Define the equivalence relation $\sim$ on S called retraction

$$x \sim y \iff \theta_x = \theta_y.$$

From the description of involutive solutions we get

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From the description of involutive solutions we get

$$\theta_{xy} = \theta_x \iff xy \sim x$$

$$x_1 \sim y_1 \text{ and } x_2 \sim y_2 \implies x_1 x_2 \sim x_1 \sim y_1 \sim y_1 y_2$$

$$\implies \sim \text{ is a congruence of } S$$

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$$\theta_{\theta_x(y)} = \theta_x\theta_y$$

$$x_1 \sim y_1 \text{ and } x_2 \sim y_2 \quad \Longrightarrow \quad \theta_{\theta_{x_1}(y_1)} = \theta_{x_1}\theta_{y_1} = \theta_{x_2}\theta_{y_2} = \theta_{\theta_{x_2}(y_2)}$$

$$\Longrightarrow \quad \theta_{x_1}(y_1) \sim \theta_{x_2}(y_2)$$

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- $\sim$ is a congruence of S
- $\theta_{x_1}(y_1) \sim \theta_{x_2}(y_2)$

This allows us to define a map $\overline{s}\colon \overline{S} \times \overline{S} \to \overline{S} \times \overline{S}$

$$\overline{s}(\overline{x}, \overline{y}) = (\overline{x}, \overline{\theta}_{\overline{x}}(\overline{y})).$$

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Hence, $\mathrm{Ret}(S, s) = (\overline{S}, \overline{s})$ is a solution of PE called retract

Irretractable involutive solutions of the PE

The retract is irretractable

- (S, s) an involutive solution of PE.

(S, s) is irretractable if $(S, s) = \mathrm{Ret}(S, s)$.

$$\begin{aligned}
\overline{\theta}_{\overline{x}} = \overline{\theta}_{\overline{y}} &\iff \overline{\theta_x(z)} = \overline{\theta_y(z)} \text{ for all } z \in S \\
&\iff \theta_{\theta_x(z)} = \theta_{\theta_y(z)} \text{ for all } z \in S \\
&\iff \theta_x\theta_z = \theta_y\theta_z \text{ for all } z \in S \\
&\iff \theta_x = \theta_y \\
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\end{aligned}$$

- $\mathrm{Ret}(S, s)$ is an irretractable involutive solution of the PE.

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- $\mathrm{Ret}(S,s)$ is an irretractable involutive solution of the PE.

Irretractable involutive solutions of the PE

The irretractable soltion is “unique”

$(A,+)$ elementary abelian 2-group

$t_A(x,y) = (x, x+y)$ $(*)$

is a irretractable involutive solution of the PE

$s_A(x,y) = (x+y, y)$

$t_A = \tau s_A \tau$

If (S,s) is an irretractable solution then $\exists +$ $(S,+)$ is an elementary abelian 2-group

$(S,s) = (S, t_S)$

Involutive solutions of PE

The extensions

- $(A, +)$ an elementary abelian 2-group
- (A, t_A) the irretractable involutive solution of the PE on A
- X a non-empty set
- $\sigma\colon A \to \mathrm{Sym}(X)$
- $S = X \times A$

Define on $S \times S$

$$s((x, a), (y, b)) = ((x, a), (\sigma_{a+b}\sigma_b^{-1}(y), a + b)).$$

- (S, s) is an involutive solution of the PE.
- $\mathrm{Ret}(S, s) = (A, t_A)$

Such a solution (S, s) is called extension of (A, t_A) by X and σ and denoted by $\mathrm{Ext}_X^\sigma(A, t_A)$.

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Solutions as extensions

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- (S, s) an involutive solution of the PE
- $\exists (A, +)$ an elementary abelian 2-group
- $\exists X$ an non-empty set
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s.t $S = X \times A$ and $(S, s) = \mathrm{Ext}_X^{\sigma}(A, t_A)$

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Involutive solutions of PE

A description

- (S, s) an involutive solution of the PE.
- $\exists A, G$ elementary abelian 2-groups
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- $\sigma \colon A \to \mathrm{Sym}(X)$ a map

s.t. $S = X \times A \times G$ and

$$(S, s) = \mathrm{Ext}_X^{\sigma}(A, t_A) \times (G, s_G)$$

- (A, t_A) is the unique irretractable invulutive solution of PE on A
- (G, s_G) is the unique bijective solution of PE on G.

Moreover, $\mathrm{Ret}(S, s) = (A, t_A)$

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Involutive solutions of PE

Isomorphic extensions

When are two extensions isomorphic as solutions?

- (S, s) and (S', s') solutions of PE
- $f : S \to S'$ a map

f is an isomorphism if f is bijective and $(f \times f)s = s'(f \times f)$

- $(A, +)$ an elementary abelian 2-group
- X a non-empty set
- $\sigma : A \to \mathrm{Sym}(X)$ and $\rho : A \to \mathrm{Sym}(X)$ maps

Then $\mathrm{Ext}_X^{\sigma}(A, t_A)$ and $\mathrm{Ext}_X^{\rho}(A, t_A)$ are isomorphic.

$$\downarrow$$

$$\{\text{involutive solutions}\} \xleftrightarrow{\text{bijective}} \left\{ X \times A \times G \;\middle|\; \begin{array}{l} X \neq \emptyset \\ A, G \text{ elem. ab. 2-group} \end{array} \right\}$$

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- $f : S \to S'$ a map

f is an isomorphism if f is bijective and $(f \times f)s = s'(f \times f)$

- $(A, +)$ an elementary abelian 2-group
- X a non-empty set
- $\sigma\colon A \to \mathrm{Sym}(X)$ and $\rho\colon A \to \mathrm{Sym}(X)$ maps

Then $\mathrm{Ext}_X^{\sigma}(A, t_A)$ and $\mathrm{Ext}_X^{\rho}(A, t_A)$ are isomorphic.

$$\downarrow$$

$$\{\text{involutive solutions}\} \xleftrightarrow{\text{bijective}} \left\{ X \times A \times G \;\middle|\; \begin{array}{l} X \neq \emptyset \\ A, G \text{ elem. ab. 2-group} \end{array} \right\}$$

Thanks for your attention!