UNIVERSITY OF EXETER

YB-semitrusses with associated bijective solutions

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Overview

Motivation

The associated solution to a YB-semitruss

- When it is bijective?

Left cancellative YB-semitruss

- An example: Skew braces

The main result

The permutation YB-semitruss

Solutions of the Yang-Baxter equation

A set-theoretic solution (of the YBE) is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$$

Write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

- r is left (resp. right) non-degenerate if λ_x (resp. ρ_x) is bijective, for any $x \in X$.
- non-degenerate if it is both left and right non-degenerate.

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Motivation

- Study arbitrary left non-degenerate solutions via an associative structure
- Consider a subclass of semitrusses called a YB-semitruss.

Definition (Brzeziński, 2018)

A semitruss is a quadruple $(A, +, \circ, \lambda)$ s.t.

- $(A, +)$ and $(A, \circ)$ are non-empty semigroups
- $\lambda : A \to \mathrm{Map}(A, A), a \mapsto \lambda_a$ is a mapping s.t.

$$\forall a, b, c \in A, \quad a \circ (b + c) = a \circ b + \lambda_a(c)$$

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There is no natural construction to associate a solution to an arbitrary semitruss.

To solve this for all left non-degenerate solutions define:

Definition

A tuple $(A, +, \circ, \lambda, \sigma)$ is a YB-semitruss if

- $(A, +, \circ, \lambda)$ is a semitruss
- $\sigma : A \to \mathrm{Map}(A, A)$, $a \mapsto \sigma_a$ is a mapping

s.t

- $\lambda_a \in \mathrm{Aut}(A, +)$ and $\lambda_a \lambda_b = \lambda_{a \circ b}$
- $a + \lambda_a(b) = a \circ b$
- $a + b = b + \sigma_b(a)$
- $\sigma_a \in \mathrm{End}(A, +)$ and $\sigma_{a+b} = \sigma_b \sigma_a$
- $\sigma_{\lambda_a(c)} \lambda_a(b) = \lambda_a \sigma_c(b)$

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The associated solution to a YB-semitruss

$(A, +, \circ, \lambda, \sigma)$ a YB-semitruss.

Define $\rho : A \to \mathrm{Map}(A, A)$, $a \mapsto \rho_a$ with

$$\forall a, b \in A, \quad \rho_b(a) = \lambda^{-1}_{\lambda_a(b)} \sigma_{\lambda_a(b)}(a)$$

Theorem

$(A, +, \circ, \lambda, \sigma)$ a YB-semitruss. Then

$$r_A : A^2 \to A^2, (a, b) \mapsto (\lambda_a(b), \rho_b(a))$$

is a solution.
Moreover

$$r_A = \varphi^{-1} s_A \phi$$

where $s_A(a, b) = (b, \sigma_b(a))$ and $\varphi(a, b) = (a, \lambda_a(b))$

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The associated solution to a YB-semitruss

When it is bijective?

$(A, +, \circ, \lambda, \sigma)$ a YB-semitruss.
r_A is bijective $\iff$ s_A is bijective $\iff$ $\forall a \in A$, σ_a is bijective

The associated solution to a YB-semitruss

The opposite YB-semitruss

Theorem

$(A, +, \circ, \lambda, \sigma)$ a YB-semitruss with σ_a bijective ($\forall a \in A$). Then

$$r_A^{-1}(a, b) = (\sigma_a^{-1}\lambda_a(b), \lambda^{-1}_{\sigma_a^{-1}\lambda_a(b)}(a))$$

Moreover, $(A, +^{op}, \circ, \bar{\lambda}, \bar{\sigma})$ is a YB-semitruss with $\bar{\lambda}_a = \sigma_a^{-1}\lambda_a$ and $\bar{\sigma}_a = \sigma_a^{-1}$.
Its associated solution is r_A^{-1}.

Left cancellative YB-semitruss

- $(A,+,\circ,\lambda)$ a semitruss $\longleftarrow$ $(A,+)$ and $(A,\circ)$ semigroups
  $\exists\lambda : A \to \mathrm{Map}(A,A)$ s.t.
  $a\circ(b+c) = a\circ b + \lambda_a(c)$.

If
- $(A,+)$ is left cancellative
- λ_a bijective $\forall a \in A$
- $a\circ b = a + \lambda_a(b)$
- $\forall a, b \in A$, $\exists x \in A$ s.t. $a\circ b = \lambda_a(b)\circ x$

Then there exists a unique anti-homomorphism $\sigma : (A,+) \to \mathrm{End}(A,+)$ s.t. $(A,+,\circ,\lambda,\sigma)$ is a YB-semitruss.

Moreover, set $\rho_b(a) = x$ for all $a, b \in A$, then

$$\sigma_b(a) = \lambda_b \rho_{\lambda_a^{-1}(b)}(a)$$

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Moreover, set $\rho_b(a) = x$ for all $a, b \in A$, then

$$\sigma_b(a) = \lambda_b \rho_{\lambda_a^{-1}(b)}(a)$$

Left cancellative YB-semitruss

An example: Skew braces

- $(A, +, \circ)$ a skew brace $\longleftarrow$ $(A, +)$ and $(A, \circ)$ groups
  $a \circ (b + c) = a \circ b \underbrace{- a + a \circ c}_{\lambda_a(c)}$

Note that

- $(A, +)$ is left cancellative
- $a + \lambda_a(b) = a \circ b$
- put $\rho_b(a) = \overline{(\bar{a} + b)} \circ b$, then $\lambda_a(b) \circ \rho_b(a) = a \circ b$

Then there exists σ such that $(A, +, \circ, \lambda, \sigma)$ is a left cancellative YB-semitruss.
In particular, it is easy to see that $\sigma_b(a) = -b + a + b$.

Left cancellative YB-semitruss

Another example

- $A = \{1, 2, 3, 4\}$
- Define $a + b = b \quad \rightsquigarrow (A, +)$ left cancellative
- $\lambda_1 = \lambda_2 = \text{id}$ and $\lambda_3 = \lambda_4 = (1\ 3)(2\ 4)$
- $a \circ b = \lambda_a(b)$
- $\sigma_b(a) = b$

It is easy to verify that $(A, +, \circ, \lambda, \sigma)$ is a left cancellative YB-semitruss. However, $(A, \circ)$ is not a group indeed $1 \circ 1 = 1 = 2 \circ 1$.

The main result

Theorem

If (X, r) is a finite left non-degenerate solution, then

$$r \text{ is bijective} \iff (X, r) \text{ is right non-degenerate}$$

The implication $\Rightarrow$ has been proven by Castelli, Catino and Stefanelli. Their proof is based on the notion of q-cycle sets (an algebraic structure introduced by Rump).

The implication $\Leftarrow$ is based on the fact that the structure monoid is a YB-semitruss.

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q-cycle sets

A set X with two binary operations $\cdot$ and $:$ is a q-cycle set if the function $X \to X, y \mapsto x \cdot y$ is bijective, and

- $(x \cdot y) \cdot (x \cdot z) = (y : x) \cdot (y \cdot z)$
- $(x : y) : (x : z) = (y \cdot x) : (y : z)$
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There is a one-to-one connection between left non-degenerate solutions and q-cycle sets.

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$\mathbb{N}$-graded YB-semitrusses

Definition

A YB-semitruss $(A, +, \circ, \lambda, \sigma)$ is (strongly) $\mathbb{N}$-graded if $A = \bigcup_{n \in \mathbb{N}} A_n$, a disjoint union of subsets so that

- $A_n + A_m \subseteq A_{n+m}$ (respectively $A_n + A_m = A_{n+m}$)
- all maps λ_a and σ_a are degree preserving.

In particular, if A is strongly $\mathbb{N}$-graded then the subsemigroup $A \setminus A_0$ is additively (and also multiplicatively) generated by its elements of degree 1.

The structure YB-semitruss

Recall that the structure monoid

$$M = M(X, r) = \langle x \in X \mid x \circ y = \lambda_x(y) \circ \rho_y(x),\ x, y \in X \rangle$$

of a left non-degenerate solution $r(x, y) = (\lambda_x(y), \rho_y(x))$ has a natural structure of YB-semitruss.

Moreover, $M(X, r)$ has a natural strongly $\mathbb{N}$-gradation (its generators $x \in X$ are the elements of degree 1),

Hence, any result on $\mathbb{N}$-graded YB-semitruss with all homogeneous components A_n finite can be translate in result on finite left non-degenerate solutions.

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Hence, any result on $\mathbb{N}$-graded YB-semitruss with all homogeneous components A_n finite can be translate in result on finite left non-degenerate solutions.

What was known?

- Any finite involutive left non-degenerate solution is non-degenerate.

  [Rump, 2005]

  [Jespers and Okniński, 2005]

- Any non-degenerate solution such that $\lambda_x = \lambda_y$ implies $x = y$ is bijective.

  [Cedó, Jespers and Verwimp, 2021]

- Any finite bijective left non-degenerate solution is right non-degenerate

  [Castelli, Catino, Stefanelli, 2021]

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The main result

key ideas

If (X, r) is a left non-degenerate solution, the diagonal map is the map

$$\mathfrak{q} : X \to X, x \mapsto \lambda_x^{-1}(x)$$

- If $(A, +, \circ, \lambda, \sigma)$ is a non-degenerate YB-semitruss, then $\mathfrak{q}$ is injective.
- $(A, +, \circ, \lambda, \sigma)$ is a non-degenerate YB-semitruss. If $\mathfrak{q}$ is bijective then r_A is bijective.
- Hence, for $(A, +, \circ, \lambda, \sigma)$ an $\mathbb{N}$-graded YB-semitruss with all homogeneous components A_n finite, if (A, r_A) is right non-degenerate then r_A is bijective.

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- Hence, for $(A, +, \circ, \lambda, \sigma)$ an $\mathbb{N}$-graded YB-semitruss with all homogeneous components A_n finite, if (A, r_A) is right non-degenerate then r_A is bijective.

The converse is based on

- For any **bijective** solution (X, r), $\mathfrak{q}$ is bijective if and only if (X, r) is non-degenerate

[Rump, 2021]

And follows the same idea used by Castelli, Catino and Stefanelli to prove that for a finite bijective left non-degenerate solution the diagonal map $\mathfrak{q}$ is bijective.

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Some remarks

It's worth mentioning that the result

- If $(A, +, \circ, \lambda, \sigma)$ is a non-degenerate YB-semitruss, then $\mathfrak{q}$ is injective.

shows how the YB-semitruss allows one to clearly distinct properties that depends on the bijectivity form properties based on non-degeneracy.

Moreover, consider the left non-degenerate solution $r(x, y) = (y, y)$. It is easy to see that $\mathfrak{q}(x) = x$. This shows that we can have $\mathfrak{q}$ bijective even if r is neither bijective nor right non-degenerate.

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Non-degenerate YB-semitruss

$(A, +, \circ, \lambda, \sigma)$ a non-degenerate YB-semitruss. We say that A is non-degenerate if the associated solution r_A is non-degenerate.

Non-degenerate YB-semitruss

The permutation YB-semitruss

$(A,+,\circ,\lambda,\sigma)$ a YB-semitruss. The set

$$\mathcal{G}(A)=\{(\sigma_a,\lambda_a,\rho_a)\mid a\in A\}$$

can be endowed with a YB-semitruss provided A is non-degenerate.

In particular, if A is non-degenerate, the map $f : A \to \mathcal{G}(A)$, $a \mapsto (\sigma_a,\lambda_a,\rho_a)$ is a YB-semitruss epimorphism.

In this case, the natural mapping $\mathcal{G}(A)\to\{(\lambda_a,\rho_a)\mid a\in A\}$ is bijective and thus the latter can be considered as the permutation YB-semitruss.

Non-degenerate YB-semitruss

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The permutation YB-semitruss

$(A, +, \circ, \lambda, \sigma)$ a non-degenerate YB-semitruss

- $\mathcal{G}(A)$ is a YB-semitruss with
  - $(\mathcal{G}(A), \circ)$ a cancellative monoid (and thus also $(\mathcal{G}(A), +)$ is left cancellative)
  - $(\mathcal{G}(A), \circ)$satisfies the left and right Ore condition.
- If, for every $a \in A$, there exists $b \in A$ such that $\lambda_a \lambda_b = \text{id}$ and $\rho_{a \circ b} = \rho_b \rho_a = \text{id}$, then the associated solution r_A is bijective and $\mathcal{G}(\mathcal{A})$ is a skew left brace.

  (for example if all λ_a and ρ_a are of finite order)
- If the subsemigroup $\langle (\lambda_x, \rho_x) \mid x \in X \rangle$ of the group $(\text{Sym}(X), \circ) \times (\text{Sym}(X), \circ^{\text{op}})$ is a group itself then r is bijective.

The permutation YB-semitruss

$(A, +, \circ, \lambda, \sigma)$ a non-degenerate YB-semitruss

- $\mathcal{G}(A)$ is a YB-semitruss with
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