The matched product of the solutions of the Yang-Baxter equation

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Noncommutative and non-associative structures, braces and applications
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The main results of this talk are conteined in

F. Catino, I.C., P. Stefanelli, *The matched product of set-theoretical solutions of the Yang-Baxter equation*, in preparation.

The Yang-Baxter equation

If X is a non-empty set, a (set-theoretical) **solution** of the Yang-Baxter equation $r : X \times X \to X \times X$ is a map such that the well-known **braid equation**

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

is satisfied, where $r_1 = r \times \mathrm{id}_X$ and $r_2 = \mathrm{id}_X \times r$.

Problem

How to obtain and construct all solutions of the Yang-Baxter equation?

Determining all set-theoretic solutions of the Yang-Baxter equation is a very difficult task. Even if we can find several works about this topic, it is still an open problem.

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Solutions of the Yang-Baxter equation

In particular, if X is a set, $r : X \times X \to X \times X$ is a solution and $a, b \in X$, then we denote

$$r(a,b) = (\lambda_a(b), \rho_b(a)),$$

where λ_a, ρ_b are maps from X into itself.

We say that r is

- **left** (resp. right) **non-degenerate** if λ_a (resp. ρ_a) is bijective, for every $a \in X$;
- **idempotent** $r^2(a,b) = r(a,b)$, for all $a, b \in X$
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Briefly, the state-of-the-art (I)

In 1999 Etingof, Schedler and Soloviev, Gateva-Ivanova and Van den Bergh laid the groundwork for studying non-degenerate involutive solutions, mainly in group theory terms.
Many results are obtained for this class by several authors.

In 2000, Lu, Yan and Zhu and independently Soloviev started to study non-degenerate solutions not necessarily involutive. In 2017, Guarnieri and Vendramin obtained new results in this context.

Finding and studying algebraic structures strictly linked with solutions is a widely used strategy to answer the question of obtaining new solutions. Although interesting and remarkable results on classifying solutions have been presented, there are still many open related problems.

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Briefly, the state-of-the-art (II)

In 1999 Etingof, Schedler, and Soloviev introduced the extensions of two involutive solutions (X, r_X) and (Y, r_Y). In particular they obtain a new solution on the union of the sets X and Y.

Gateva-Ivanova and Majid (2008) improved this result by regular extension and they found a one-to-one correspondence between regular extensions and regular pairs of actions. Given two involutive solution (X, r_X) and (Y, r_Y) they introduce another way to obtain a new solution over $X \cup Y$, the strong twisted unions.

Recently, Bachiller and Cedó obtain a new method to construct involutive non-degenerate solutions $\left(X^n, r^{(n)}\right)$ of the Yang-Baxter equation, for any positive integer n, from a given solution (X, r).

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Solution: a characterization

Let X be a non-empty set and $r : X \times X \to X \times X$ a map. If λ_x and ρ_x, for every $x \in X$ are maps such that $r(x, y) = (\lambda_x(y), \rho_y(x))$ for all $x, y \in X$ then (X, r) is a solution if and only if the following properties hold:

1. $\lambda_x\lambda_y = \lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}$, for all $x, y \in X$;
2. $\rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y) = \lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y)$, for all $x, y, z \in X$;
3. $\rho_z\rho_y = \rho_{\rho_z(y)}\rho_{\lambda_y(z)}$, for all $y, z \in X$.

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3. $\rho_z \rho_y = \rho_{\rho_z(y)} \rho_{\lambda_y(z)}$, for all $y, z \in X$.

Definition: the matched product system

Let (S, r_s) and (T, r_T) be solutions and $\alpha : T \to \mathrm{Sym}(S)$, $\beta : S \to \mathrm{Sym}(T)$ maps, put $\alpha(u) := \alpha_u$, for every $u \in T$ and $\beta(a) := \beta_a$, for every $a \in S$. If S, r_S, T, r_T, α and β satisfy the following conditions

$$\alpha_u \alpha_v = \alpha_{\lambda_u(v)} \alpha_{\rho_v(u)}; \qquad \beta_a \beta_b = \beta_{\lambda_a(b)} \beta_{\rho_b(a)};$$

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for all $u, v \in T$ and $a, b \in S$, then we call $(S, r_S, T, r_T, \alpha, \beta)$ a **matched product system of solutions**.

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for all $u, v \in T$ and $a, b \in S$, then we call $(S, r_S, T, r_T, \alpha, \beta)$ a **matched product system of solutions**.

Theorem: the matched product of solutions

Let $(S, r_S, T, r_T, \alpha, \beta)$ be a matched product system. If we set

$$\lambda_{(a,u)}(b,v) := \left(\alpha_u \lambda_{\alpha_u^{-1}(a)}(b),\ \beta_a \lambda_{\beta_a^{-1}(u)}(v)\right)$$

$$\rho_{(b,v)}(a,u) :=$$

$$\left(\alpha^{-1}_{\beta^{-1}_{\alpha_u \lambda_{\alpha_u^{-1}(a)}(b)} \beta_a \lambda_{\beta_a^{-1}(u)}(v)} \rho_{\alpha_{\beta_a^{-1}(u)}(b)}(a),\ \beta^{-1}_{\alpha^{-1}_{\beta_a \lambda_{\beta_a^{-1}(u)}(v)} \alpha_u \lambda_{\alpha_u^{-1}(a)}(b)} \rho_{\beta_{\alpha_u^{-1}(a)}(v)}(u)\right),$$

for all $a, b \in S$ and $u, v \in T$, then the map $r : S{\times}T \times S{\times}T \to S{\times}T \times S{\times}T$ defined by

$$r\left((a,u),(b,v)\right) := \left(\lambda_{(a,u)}(b,v),\ \rho_{(b,v)}(a,u)\right),$$

for all $a, b \in S$ and $u, v \in T$, is a solution that we call the **matched product solution** of r_S and r_T (via α and β), denoted by $r_S \bowtie r_T$.

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for all $a, b \in S$ and $u, v \in T$, is a solution that we call the **matched product solution** of r_S and r_T (via α and β), denoted by $r_S \bowtie r_T$.

Theorem: the matched product of solutions

Let $(S, r_S, T, r_T, \alpha, \beta)$ be a matched product system. If we set

$$\lambda_{(a,u)}(b,v) := \left(\alpha_u \lambda_{\alpha_u^{-1}(a)}(b),\ \beta_a \lambda_{\beta_a^{-1}(u)}(v)\right)$$

$$\rho_{(b,v)}(a,u) :=$$

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Particular case

A characterization of involutive left non-degenerate solution

Let X be a non-empty set and $r : X \times X \to X \times X$ a map. Indicate the image $r(x, y) := (\lambda_x(y), \rho_y(x))$ for all $x, y \in X$, where $\lambda_x, \rho_x : X \to X$ are maps. (X, r) is a left non-degenerate involutive solution if and only if the following properties hold:

1. $\lambda_x \in \operatorname{Sym}(X)$, for every $x \in X$;
2. $\rho_y(x) = \lambda^{-1}_{\lambda_x(y)}(x)$, for all $x, y \in X$;
3. $\lambda_x \lambda_{\lambda^{-1}_x(y)} = \lambda_y \lambda_{\lambda^{-1}_y(x)}$, for all $x, y \in X$.

Particular case

The matched product of left non-degenerate involutive solutions (II)

Let (S, r_S), (T, r_T) be left non-degenerate involutive solution and $\alpha : T \to \mathrm{Sym}(S)$, $\beta : S \to \mathrm{Sym}(T)$ maps that satisfy

$$\alpha_u \alpha_{\lambda_u^{-1}(v)} = \alpha_v \alpha_{\lambda_v^{-1}(u)} \qquad \beta_a \beta_{\lambda_a^{-1}(b)} = \beta_b \beta_{\lambda_b^{-1}(a)}$$

$$\lambda_a \alpha_{\beta_a^{-1}(u)} = \alpha_u \lambda_{\alpha_u^{-1}(a)} \qquad \lambda_u \beta_{\alpha_u^{-1}(a)} = \beta_u \lambda_{\beta_a^{-1}(u)}$$

for all $a, b \in S$ and $u, v \in T$. Then $(S, r_S, T, r_T, \alpha, \beta)$ is a matched product system. In particular, in this case the conditions

$$\rho_{\alpha_u^{-1}(b)} \alpha^{-1}_{\beta_a(u)}(a) = \alpha^{-1}_{\beta_{\rho_b(a)} \beta_b^{-1}(u)} \rho_b(a) \quad \rho_{\beta_a^{-1}(v)} \beta^{-1}_{\alpha_u(a)}(u) = \beta^{-1}_{\alpha_{\rho_v(u)} \alpha_v^{-1}(a)} \rho_v(u)$$

are satisfied.

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Particular case

The matched product of left non-degenerate involutive solutions (III)

Let (S, r_S), (T, r_T) be left non-degenerate involutive solution and $\alpha : T \to \mathrm{Sym}\,(S)$, $\beta : S \to \mathrm{Sym}\,(T)$ maps that satisfy

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for all $a, b \in S$ and $u, v \in T$. Then $(S, r_S, T, r_T, \alpha, \beta)$ is a matched product system and matched product solution of r_S and r_T is left non-degenerate and involutive. In particular, with respect to the same definition of $\lambda_{(a,u)} : S \times T \to S \times T$, i.e,

$$\lambda_{(a,u)}(b, v) := \left(\alpha_u \lambda_{\alpha_u^{-1}(a)}(b),\ \beta_a \lambda_{\beta_a^{-1}(u)}(v)\right),$$

we have that the matched product solution is the map $r : S \times T \times S \times T \to S \times T \times S \times T$ given by

$$r\left((a, u), (b, v)\right) := \left(\lambda_{(a,u)}(b, v),\ \lambda^{-1}_{\lambda_{(a,u)}(b,v)}(a, u)\right),$$

for all $a, b \in S$, $u, v \in T$.

Particular case

The matched product of left non-degenerate involutive solutions (III)

Let (S, r_S), (T, r_T) be left non-degenerate involutive solution and $\alpha : T \to \operatorname{Sym}(S)$, $\beta : S \to \operatorname{Sym}(T)$ maps that satisfy

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An example

Let $r : S \times S \to S \times S$ be an involutive left non-degenerate solution. If $\alpha, \beta : S \to \mathrm{Sym}\,(S)$ are defined by $\alpha_u := \lambda_u$ and $\beta_a := \lambda_a$, for all $a, u \in S$, then $(S, r, S, r, \alpha, \beta)$ is a matched product system. In fact if satisfies

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since (S, r) is an involutive left non-degenerate solution.

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Thanks for your attention!