

The algebraic structure of semi-brace

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Braces

In 2007, Rump introduced braces in order to find set-theoretical solutions of the Yang-Baxter equation. A basic equation of statistical mechanics.

In 2015, Cedó, Jespers and Okrinski proved the following definition is equivalent to the original one and is very useful to construct new braces.

Definition

Let B be a set with two operations $+$ and \circ such that $(B, +)$ is an abelian group and (B, \circ) is a group. We say that $(B, +, \circ)$ is a **(left) brace** if

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

holds for all $a, b, c \in B$, where $-a$ is the inverse of a respect to $+$.

Note that, if $(R, +, \cdot)$ is a radical ring and if we set

$$a \circ b := a \cdot b + a + b,$$

for all $a, b \in R$, then $(R, +, \circ)$ is a left brace.

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Skew Brace

Braces allows us to construct only involutive non-degenerate solutions of the Yang-Baxter equation. In 2016, Guarnieri and Vendramin introduced a new algebraic structure, the skew braces, in order to obtain bijective solutions not necessarily involutive.

Definition

Let B be a set with two operations $+$ and \circ such that $(B, +)$ and (B, \circ) are groups. We say that $(B, +, \circ)$ is a **skew (left) brace** if

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holds for all $a, b, c \in B$, where $-a$ is the inverse of a respect to $+$.

Clearly, every brace is a skew brace. Further, if $(B, +)$ is a group and we set $a \circ b := a + b$, for all $a, b \in B$, then $(B, +, \circ)$ is a skew brace (known as **zero skew brace**), in particular if $(B, +)$ is a non-abelian group then it is a skew brace that is not a brace.

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Semi-brace

A further generalization of braces, the semi-braces, allow us to obtain solution left non-degenerate and not necessarily bijective.

Definition (F. Catino, I.C. and P. Stefanelli, J. Algebra, 2017)

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- ▶ $(B, +)$ is a group and, in particular, a left cancellative semigroup;
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Examples of semi-braces

- If (E, \circ) is a group, then $(E, +, \circ)$, where $a + b = b$, for all $a, b \in E$ is a semi-brace. In fact,
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We call this semi-brace the **trivial left semi-brace**.

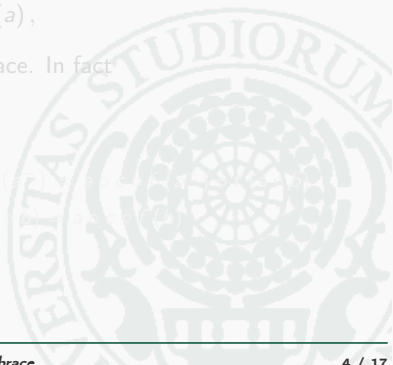
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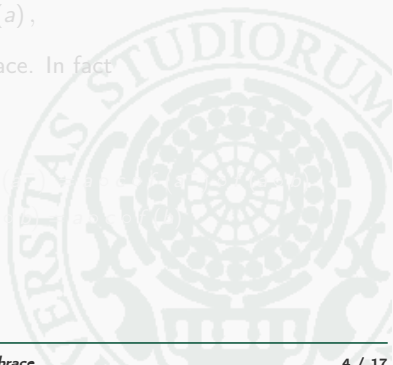
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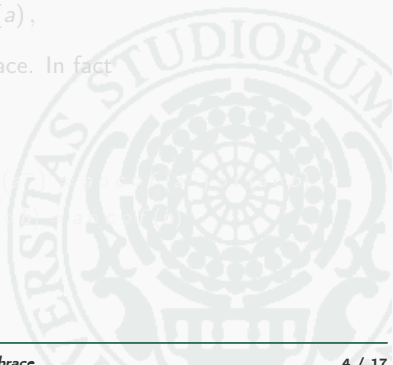
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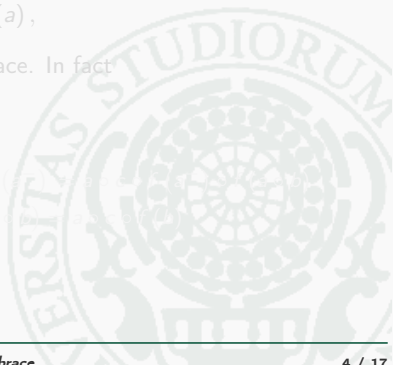
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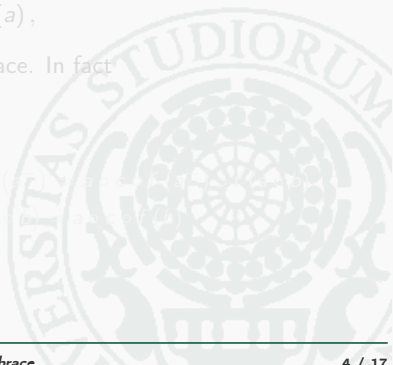
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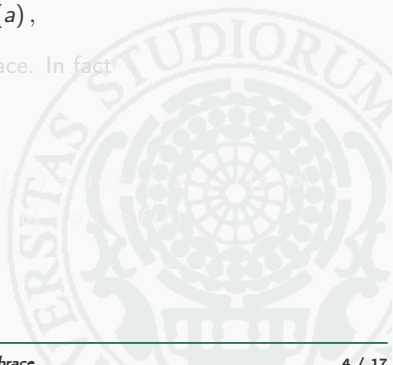
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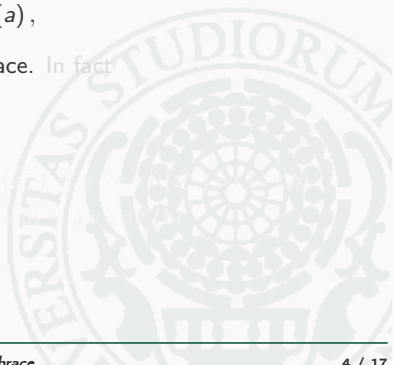
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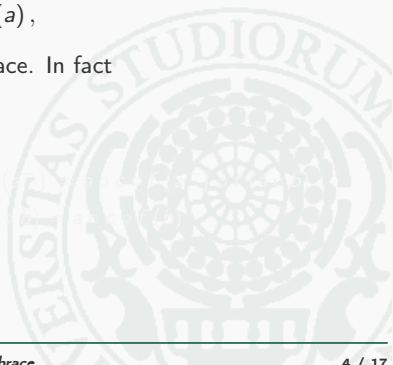
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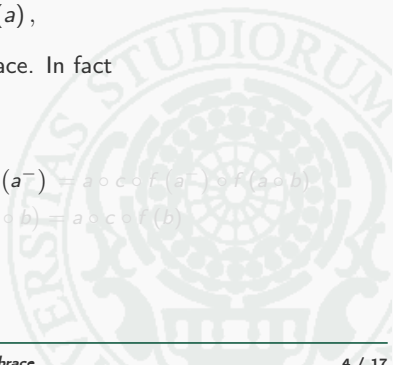
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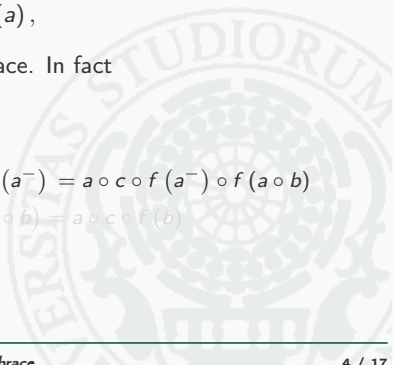
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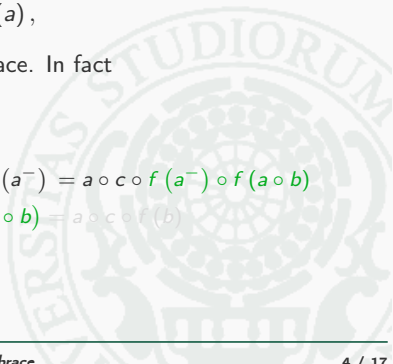
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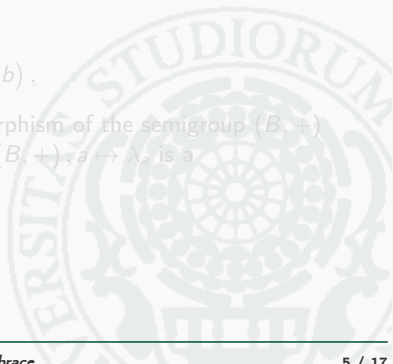
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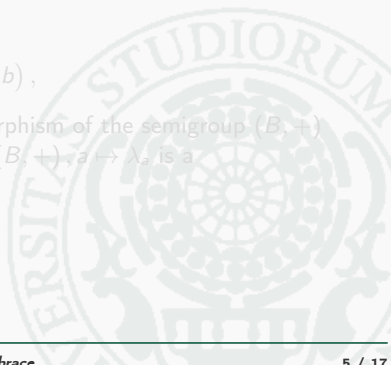
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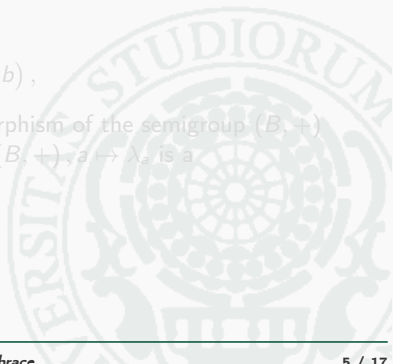
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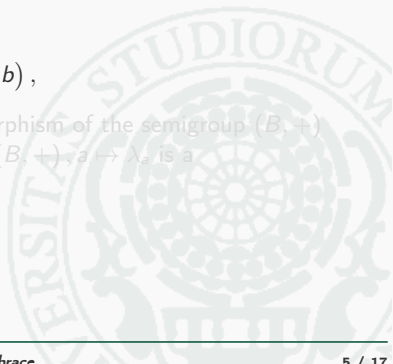
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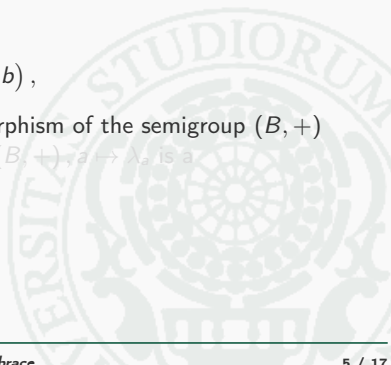
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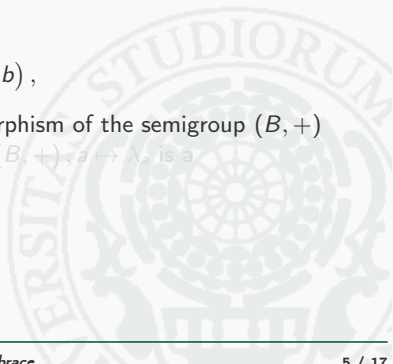
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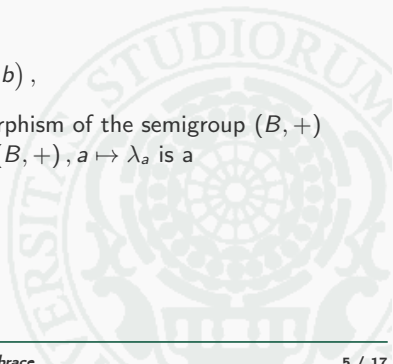
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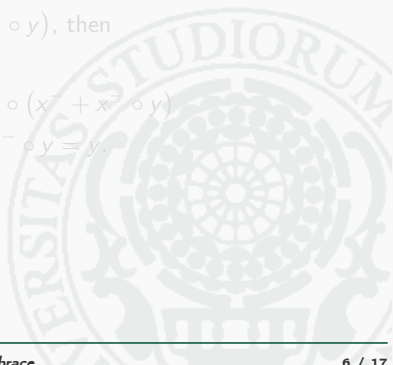
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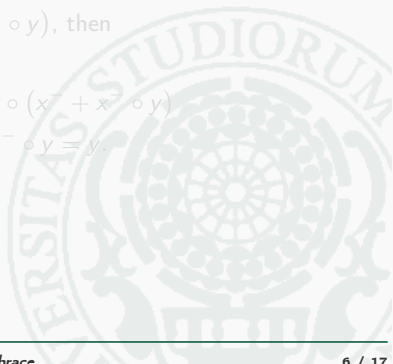
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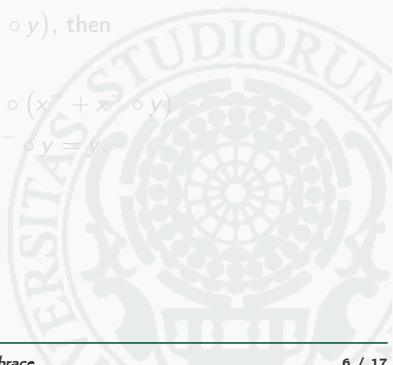
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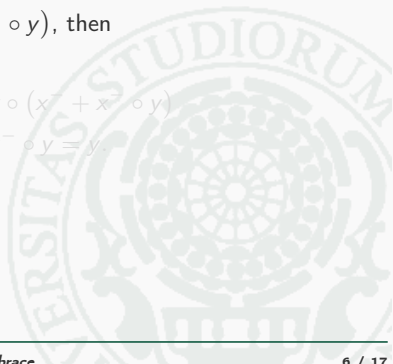
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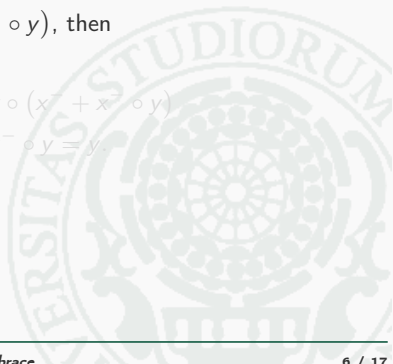
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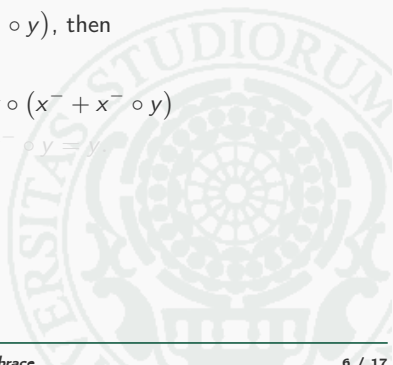
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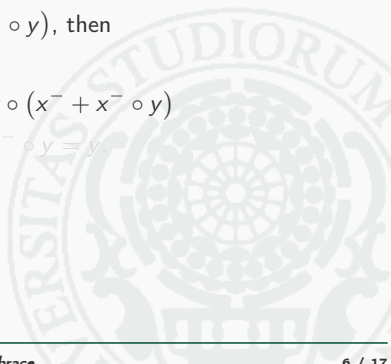
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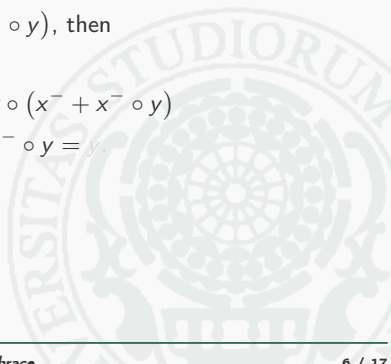
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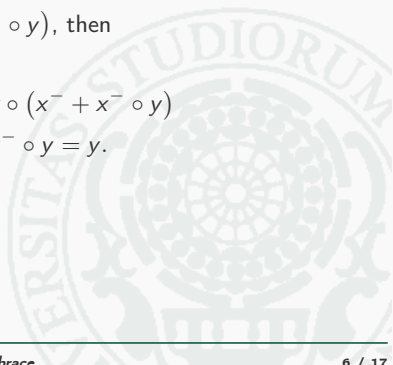
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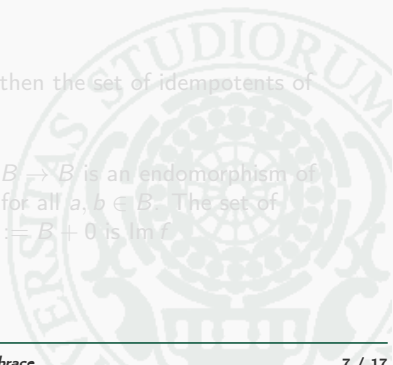
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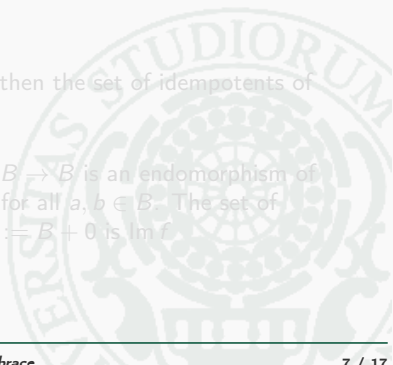
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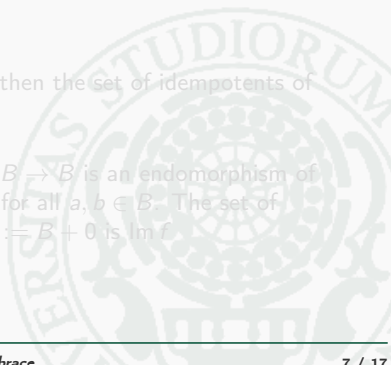
Moreover, it is well-known that if B is a right group, E is the set of idempotents, then $G_e := B + e$, for every $e \in E$, is a group and $B = G_e + E$

In particular if B is a brace and E is the set of idempotents of $(B, +)$, then 0 the identity of the group (B, \circ) lies in E . Therefore $G := B + 0$ is a group respect to the sum and

$$B = G + E.$$

For example, if $(E, +, \circ)$ is a trivial semi-brace, then the set of idempotents of $(E, +)$ is E and the group $G = \{0\}$.

Further, if $(B, +, \circ)$ is the semi-brace where $f : B \rightarrow B$ is an endomorphism of the group (B, \circ) , $f^2 = f$ and $a + b = b \circ f(a)$, for all $a, b \in B$. The set of idempotents of $(B, +)$ is $\ker f$ and the group $G := B + 0$ is $\text{Im } f$.



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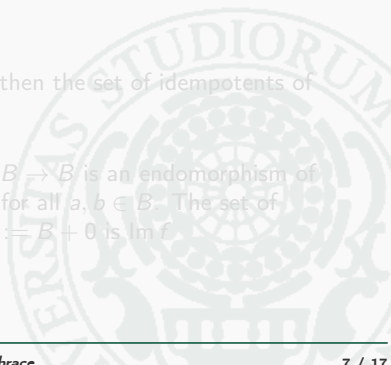
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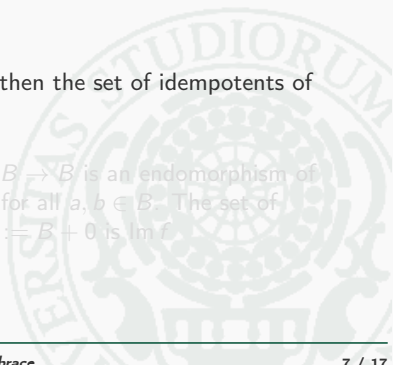
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$$\rho_b(a) := (a^- + b)^- \circ b,$$

for every $b \in B$. We have that the map $\rho : B \rightarrow B^B$ given by $\rho(b) = \rho_b$ is a semigroup antihomomorphism from the group (B, \circ) into the monoid B^B of all maps from B into itself.

Proposition (F. Catino, I. C., P. Stefanelli, J. Algebra, 2017)

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A structural theorem for semi-brace (I)

Theorem (F. Catino, I.C., P. Stefanelli, to appear)

Let G be a skew brace and E a trivial semi-brace, $\delta : G \rightarrow \text{Sym}(E)$ a right action of the group (G, \circ) on the set E and $\sigma : E \rightarrow \text{Aut}(G)$ a left action of the group (E, \circ) on the set G and σ_e is an automorphism of the group $(G, +)$, for every $e \in E$, such that

1. $e \cdot (g_1 \circ g_2) = (e \cdot g_1) \circ (e \cdot g_2)$;
2. $(e_1 \circ e_2)^g = e_1^{\sigma_2 g} \circ e_2^g$;
3. $0^e = 0$.

hold for all $g, g_1, g_2 \in G$ and $e, e_1, e_2 \in E$. Then the sum and the multiplication over the cartesian product $G \times E$ given by

$$(g_1, e_1) + (g_2, e_2) := (g_1 + g_2, e_2)$$

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Let G be a skew brace and E a trivial semi-brace, $\delta : G \rightarrow \text{Sym}(E)$ a right action of the group (G, \circ) on the set E and $\sigma : E \rightarrow \text{Aut}(G)$ a left action of the group (E, \circ) on the set G and σ_e is an automorphism of the group $(G, +)$, for every $e \in E$, such that

1. ${}^e(g_1 \circ g_2) = ({}^e g_1) \circ ({}^{e^{\delta_1}} g_2)$;
2. $(e_1 \circ e_2)^\delta = e_1^{e_2^\delta} \circ e_2^\delta$;
3. $0^\delta = 0$,

hold for all $g, g_1, g_2 \in G$ and $e, e_1, e_2 \in E$. Then the sum and the multiplication over the cartesian product $G \times E$ given by

$$(g_1, e_1) + (g_2, e_2) := (g_1 + g_2, e_2)$$

$$(g_1, e_1) \circ (g_2, e_2) := \left(g_1 \circ \left((e_1^-)^{\delta_1} \right)^-, e_1 \circ \left((e_2^-)^{\left(e_1^- g_1 \right)^-} \right)^- \right)$$

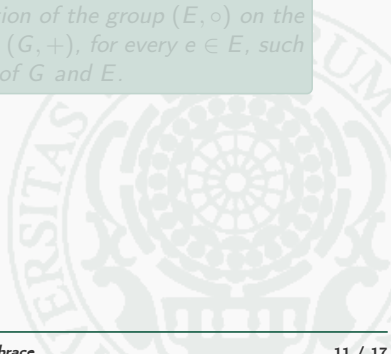
define a structure of semi-brace, known as **matched product of G and E** (via δ and σ).

A structural theorem for semi-brace (II)

Vice versa

Theorem

Let B be a semi-brace, E the set of idempotents of $(B, +)$ and $G := B + 0$. Then G is a skew brace, E is a trivial semi-brace and there exist a right action of the group (G, \circ) on the set E and a left action of the group (E, \circ) on the set G and σ_e is an automorphism of the group $(G, +)$, for every $e \in E$, such that B is isomorphic to the matched product of G and E .

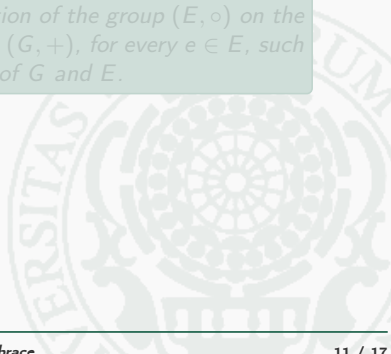


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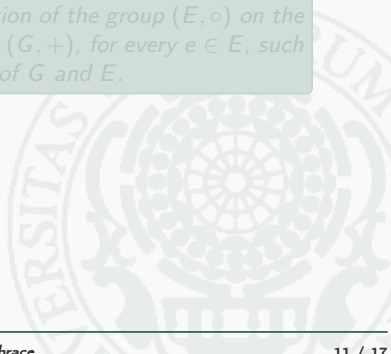


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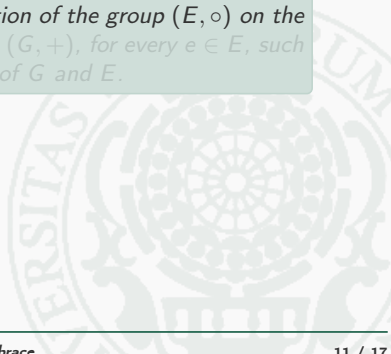


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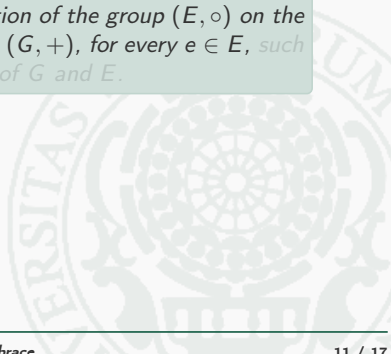


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Ideals of semi-braces (I)

Definition (F. Catino, I. C., P. Stefanelli, J. Algebra, 2017)

Let B be a semi-brace, E the set of idempotents of $(B, +)$, $G := B + 0$. We say that a subsemigroup I of $(B, +)$ is an **ideal** if

- ▶ I is a normal subgroup of (B, \circ) ;
- ▶ $I \cap G$ is a normal subgroup of $(G, +)$;
- ▶ $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$;
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B and $\{0\}$ are ideals of B that we call **trivial ideals** of B .



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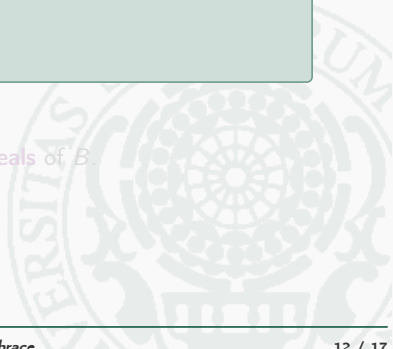
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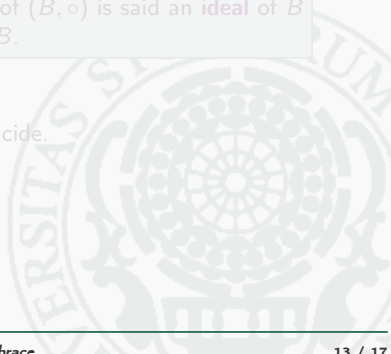
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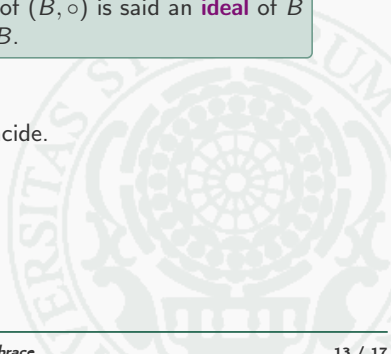
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If B is a semi-brace and I is an ideal of B , then the relation \sim_I on B given by

$$\forall x, y \in B, \quad x \sim_I y \iff y^- \circ x \in I$$

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The socle

An classical ideal introduced for braces and skew braces is the socle. We may introduce a definition of socle for semi-braces that includes the correspondent definition for braces and skew braces.

Definition (F. Catino, I.C., P. Stefanelli, J. Algebra, 2017)

If B is a semi-brace. We call the set given by

$$\text{Soc}(B) = \{a \mid a \in B, \lambda_a = \lambda_0, \rho_a = \rho_0\}$$

the **socle** of the semi-brace B .

We may prove, that if B is a semi-brace and $G := B \neq 0$ then

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Example of socle of a semi-brace

If (B, \circ) is a group and $f : B \rightarrow B$ is an endomorphism of (B, \circ) such that $f^2 = f$. We already saw that if we set $a + b := b \circ f(a)$, for every $a, b \in B$, that we have a semi-brace. In this case the socle is given by

$$\text{Soc}(B) = Z(B) \cap \text{Im } f$$

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Thank you for your attention!

