The algebraic structure of semi-brace

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UNIVERSITÀ DEL SALENTO

Young Researchers Algebra Conference 2017

Naples, May 23rd, 2017

Braces

In 2007, Rump introduced braces in order to find set-theoretical solutions of the Yang-Baxter equation. A basic equation of statistical mechanics.
In 2015, Cedó, Jespers and Okńinski proved the following definition is equivalent to the original one and is very useful to construct new braces.

Definition

Let B be a set with two operations $+$ and $\circ$ such that $(B, +)$ is an abelian group and $(B, \circ)$ is a group. We say that $(B, +, \circ)$ is a **(left) brace** if

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

holds for all $a, b, c \in B$, where $-a$ is the inverse of a respect to $+$.

Note that, if $(R, +, \cdot)$ is a radical ring and if we set

$$a \circ b := a \cdot b + a + b,$$

for all $a, b \in R$, then $(R, +, \circ)$ is a left brace.

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Skew Brace

Braces allows us to construct only involutive non-degenerate solutions of the Yang-Baxter equation. In 2016, Guarnieri and Vendramin introduced a new algebraic structure, the skew braces, in order to obtain bijective solutions not necessarily involutive.

Definition

Let B be a set with two operations $+$ and $\circ$ such that $(B, +)$ and $(B, \circ)$ are groups. We say that $(B, +, \circ)$ is a **skew (left) brace** if

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

holds for all $a, b, c \in B$, where $-a$ is the inverse of a respect to $+$.

Clearly, every brace is a skew brace. Further, if $(B, +)$ is a group and we set $a \circ b := a + b$, for all $a, b \in B$, then $(B, +, \circ)$ is a skew brace (known as **zero skew brace**), in particular if $(B, +)$ is a non-abelian group then it is a skew brace that is not a brace.

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Semi-brace

A further generalization of braces, the semi-braces, allow us to obtain solution left non-degenerate and not necessarily bijective.

Definition (F. Catino, I.C, and P. Stefanelli, J. Algebra, 2017)

Let B be a set with two operations $+$ and $\circ$ such that $(B,+)$ is a left cancellative semigroup and $(B,\circ)$ is a group. We say that $(B,+,\circ)$ is a **(left) semi-brace** if

$$a\circ(b+c) = a\circ b + a\circ\left(a^{-}+c\right),$$

holds for all $a,b,c\in B$, where a^{-} is the inverse of a respect to $\circ$.

Note that, if B is a skew brace, then it is a semi-brace. In fact

- $(B,+)$ is a group and, in particular, a left cancellative semigroup;
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Examples of semi-braces

1. If $(E, \circ)$ is a group, then $(E, +, \circ)$, where $a + b = b$, for all $a, b \in E$ is a semi-brace. In fact,
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   We call this semi-brace the **trivial left semi-brace**.

2. If $(B, \circ)$ is a group and f is an endomorphism of $(B, \circ)$ such that $f^2 = f$. Set

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1. If $(E, \circ)$ is a group, then $(E, +, \circ)$, where $a + b = b$, for all $a, b \in E$ is a semi-brace. In fact,
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Thanks to the maps λ_a we may prove that the additive semigroup of a semi-brace is a right group.

Recall that a left cancellative semigroup B is a **right group** if and only if for all $x, y \in B$ there exists $t \in B$ such that $x + t = y$.

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Moreover, it is well-known that if B is a right group, E is the set of idempotents, then $G_e := B + e$, for every $e \in E$, is a group and $B = G_e + E$

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Further, if $(B, +, \circ)$ is the semi-brace where $f : B \to B$ is an endomorphism of the group $(B, \circ)$, $f^2 = f$ and $a + b = b \circ f(a)$, for all $a, b \in B$. The set of idempotents of $(B, +)$ is $\ker f$ and the group $G := B + 0$ is $\operatorname{Im} f$

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If B is a semi-brace and $b \in B$ we may introduce the map $\rho_b : B \to B$ defined by

$$\rho_b(a) := \left(a^- + b\right)^- \circ b,$$

for every $b \in B$. We have that the map $\rho : B \to B^B$ given by $\rho(b) = \rho_b$ is a semigroup antihomomorphism from the group $(B, \circ)$ into the monoid B^B of all maps from B into itself.

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If B is a semi-brace, E the set of all idempotents of $(B, +)$ and $G := B + 0$, then

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Let B be a semi-brace, E the set of idempotents of $(B, +)$, and $G := B + 0$.

- We may prove that $(G, \circ)$ and $(E, \circ)$ are subgroups of $(B, \circ)$.
- Further, we have that $G \cap E = \{0\}$ and $B = G \circ E$.
- Hence $(B, \circ)$ is the matched product of the groups $(G, \circ)$ and $(E, \circ)$.

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Let G be a skew brace and E a trivial semi-brace, $\delta : G \to \operatorname{Sym}(E)$ a right action of the group $(G, \circ)$ on the set E and $\sigma : E \to \operatorname{Aut}(G)$ a left action of the group $(E, \circ)$ on the set G and σ_e is an automorphism of the group $(G, +)$, for every $e \in E$, such that

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Let B be a semi-brace, E the set of idempotents of $(B, +)$, $G := B + 0$. We say that a subsemigroup I of $(B, +)$ is an **ideal** if

- I is a normal subgroup of $(B, \circ)$;
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- $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$;
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B and $\{0\}$ are ideals of B that we call **trivial ideals** of B.

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Guarnieri and Vendramin give the following definition of ideal for a skew brace (that is a semi-brace with a group as additive structure)

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Proposition (F. Catino, I. C., P. Stefanelli, J. Algebra, 2017)

If B is a semi-brace and I is an ideal of B, then the relation $\sim_I$ on B given by

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is a congruence of B.

Further, if B is a semi-brace and I is an ideal, then the quotient structure B/I of B respect to the relation $\sim_I$ is a right group respect to the sum.

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The socle

An classical ideal introduced for braces and skew braces is the socle. We may introduce a definition of socle for semi-braces that includes the correspondent definition for braces and skew braces.

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the **socle** of the semi-brace B.

We may prove, that if B is a semi-brace and $G := B + 0$ then

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Example of socle of a semi-brace

If $(B, \circ)$ is a group and $f : B \to B$ is an endomorphism of $(B, \circ)$ such that $f^2 = f$. We already saw that if we set $a + b := b \circ f(a)$, for every $a, b \in B$, that we have a semi-brace. In this case the socle is given by

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Thank you for your attention!