Asymmetric product of braces

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Radical brace

Cedó, Jespers and Okńinski proved, in 2015, the following definition is equivalent to the original one introduced by Rump in 2007.

Definition

A *(radical) brace* is a set B with two operation $+$ and $\circ$ such that $(B,+)$ is an abelian group, $(B,\circ)$ is a group and

$$(a+b)\circ c + c = a\circ b + a\circ c \tag{1}$$

for all $a,b,c \in B$. We call $(B,+)$ the *additive group* and $(B,\circ)$ the *adjoint group* of the brace.

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Radical F-brace

Following Rump's terminology, a vector space V over a field F with a multiplication $\circ$ is called *(radical) F-brace* if V is a radical brace and

$$\lambda(u \circ v) = (\lambda u) \circ v + (\lambda - 1)v \tag{2}$$

for all $\lambda \in F$ and for all $u, v, w \in V$.

This is an important class of braces because there is a link between radical F-braces and regular subgroups of affine group over the same vector space (F. Catino, R. Rizzo, 2009).

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The aim of this communication is to introduce a new construction of braces, and then of F-braces, called *asymmetric product*.

The F-braces obtained by asymmetric product have adjoint group which is a generalization of subgroups obtained by Hegedüs in 2000, that have trivial intersection with the translation group.

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Hegedüs' subgroup (I)

Hegedüs constructs particular subgroup of the affine group $AGL(n, p)$ (with p odd prime and $n \geq 4$ or $p = 2$ e $n = 3$ or $n \geq 5$ odd) with trivial intersection with the translation group.

Let p be a prime and $n \in \mathbb{N}$, if p is odd, let $n \geq 4$ otherwise if $p = 2$, let $n \geq 3$ odd. If $\mathbb{F}_p$ is the field of p elements we consider

- $\mathfrak{q} : \mathbb{F}_p^{n-1} \to \mathbb{F}_p$ a non-degenerate quadratic form;
- $\mathfrak{b} : \mathbb{F}_p^{n-1} \times \mathbb{F}_p^{n-1} \to \mathbb{F}_p$ the polar form of $\mathfrak{q}$ (i.e. $(x, y)\mathfrak{b} = (x + y)\mathfrak{q} - (x)\mathfrak{q} - (y)\mathfrak{q}$, for all $x, y \in \mathbb{F}_p^{n-1}$);

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Hegedüs' subgroup (II)

- X the matrix associated with $\mathfrak{b}$ respect a fixed basis ($(v+w)\mathfrak{q} = v\mathfrak{q} + w\mathfrak{q} + vXw^T$ for all $v, w \in \mathbb{F}_p^{n-1}$);
- A a non-degenerate, orthogonal $(n-1) \times (n-1)$-matrix (i.e. $v\mathfrak{q} = (vA)\mathfrak{q}$ for all $v \in \mathbb{F}_p^{n-1}$).

Then

$$T := \left\{ \begin{pmatrix} 1 & v\mathfrak{q} + m & v \\ 0 & 1 & 0 \\ 0 & A^m X w^T & A^m \end{pmatrix} \middle| m \in \mathbb{F}_p, v \in \mathbb{F}_p^{n-1} \right\} \quad (3)$$

is the Hegedüs' subgroup of the $AGL(n, p)$.

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Asymmetric product of braces

We recall that, if S and T are abelian group, then a function $\mathfrak{b} : T \times T \to S$ is a *symmetric cocycle* on T with values in S if $(t_1, t_2)\mathfrak{b} = (t_2, t_1)\mathfrak{b}$ and,

$$(0,0)\mathfrak{b} = 0, \tag{4}$$

$$(t_1 + t_2, t_3)\mathfrak{b} + (t_1, t_2)\mathfrak{b} = (t_1, t_2 + t_3)\mathfrak{b} + (t_2.t_3)\mathfrak{b}, \tag{5}$$

hold for all $t_1, t_2, t_3 \in T$.

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Main theorem

Theorem (F. Catino, I.C., P. Stefanelli, to appear)

Let S and T be radical braces, let $\mathfrak{b} : T \times T \to S$ be a symmetric cocycle on $(T, +)$ with values in $(S, +)$, and let $\alpha : S \longrightarrow Aut(T)$ be a homomorphism of the adjoint group of S to the group of automorphisms of the radical brace T such that

$$(t_1, t_2)\mathfrak{b} \circ s + ((t_1 + t_2)^s \circ t_3, t_3)\mathfrak{b} = (t_1^s \circ t_3, t_2^s \circ t_3)\mathfrak{b} + s, \tag{6}$$

holds for all $s \in S$ and $t_1, t_2, t_3 \in T$. Then, the addition and the multiplication over $S \times T$ given by

$$(s_1, t_1) + (s_2, t_2) = (s_1 + s_2 + (t_1, t_2)\mathfrak{b},\ t_1 + t_2) \tag{7}$$

$$(s_1, t_1) \circ (s_2, t_2) = (s_1 \circ s_2,\ t_1^{s_2} \circ t_2) \tag{8}$$

define a structure of radical brace on $S \times T$.

We call this radical brace the *asymmetric product* of T by S (via $\mathfrak{b}$ and α) denoted by $S \ltimes_\circ T$.

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Asymmetric product of F-braces (I)

As we expect, in general, the asymmetric product between two F-braces gives a brace but it does not give an F-brace.

Example (I)

Let $\mathbb{F}_2$ be the field with 2 elements and $S := \mathbb{F}_2$ and $T := \mathbb{F}_2 \times \mathbb{F}_2$ vector spaces over $\mathbb{F}_2$. We can see S and T as zero $\mathbb{F}_2$-braces defining the operations $\circ$ such that $s_1 \circ s_2 = s_1 + s_2$ and $t_1 \circ t_2 = t_1 + t_2$, for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$ respectively. Set

$$\mathfrak{b} : T \times T \to S, \quad ((b_1, c_1), (b_2, c_2)) \mapsto b_1 b_2 + c_1 c_2 \tag{9}$$

and

$$\alpha : S \to Aut(T) \tag{10}$$

such that $0\alpha = id_T$ and $1\alpha : T \to T, (b_1, c_1) \mapsto (c_1, b_1)$. Then $S, T, \mathfrak{b}, \alpha$ satisfy the properties of the theorem and so $S \ltimes_\circ T$ is a brace.

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Asymmetric product of F-braces (II)

The asymmetric product of S and T is not an $\mathbb{F}_2$-brace. In fact, the element $(1,0,1)$ has order 4 in $(S \ltimes_\circ T, +)$, $(1,0,1)+(1,0,1)=(1,0,0)$, $(1,0,0)+(1,0,1)=(0,0,1)$ and $(0,0,1)+(1,0,1)=(0,0,0)$

Asymmetric product of F-braces - odd characteristic

We have to require another condition to obtain an F-brace from F-braces.

Theorem

Let F be a field of characteristic $p \neq 2$. Let S and T be radical F-braces, let $\mathfrak{b} : T \times T \to S$ be a bilinear and symmetric map and, let $\alpha : S \longrightarrow Aut(T)$ be a homomorphism of the multiplicative group of S to the group of automorphisms of the radical F-brace T that satisfy the condition

$$(t_1, t_2)\mathfrak{b} \circ s + ((t_1 + t_2)^s \circ t_3, t_3)\mathfrak{b} = (t_1^s \circ t_3, t_2^s \circ t_3)\mathfrak{b} + s, \tag{11}$$

for all $s \in S$ and $t_1, t_2, t_3 \in T$.

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Asymmetric product of F-braces - odd characteristic

Then the asymmetric product $S \ltimes_\circ T$ is a radical F-brace with the scalar multiplication given by

$$\lambda(s,t) = \left(\lambda s + \frac{\lambda(\lambda-1)}{2}(t,t)\mathfrak{b}, \lambda t\right) \tag{12}$$

for all $\lambda \in F$, $s \in S$ and $t \in T$.

Observe that the condition (11) is the same as that one in the main theorem.

Asymmetric product of F-braces - even characteristic

If the characteristic of the field F is 2 then the condition (11) in previous proposition is not sufficient for having a structure of F-brace.

Theorem

Let V be a vector space over a field F of characteristic 2, let $\mathfrak{q}$ be a quadratic form on V, let $\mathfrak{b}_\mathfrak{q}$ be the polar form of $\mathfrak{q}$ and $\alpha : F \to Aut(V)$ be an homomorphism of the adjoint group of F to the group of automorphism of the radical F-brace V. If it is satisfied

$$(v_1^s \circ v_2, v_2)\,\mathfrak{b}_\mathfrak{q} = (v_1)\,\mathfrak{q} \circ s + s + (v_2)\,\mathfrak{q} + (v_1^s \circ v_2)\,\mathfrak{q} \qquad (13)$$

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Then the asymmetric product $F \ltimes_\circ V$ is a radical F-brace with scalar multiplication given by

$$\lambda(s_1, v_1) = (\lambda s_1 + \lambda(\lambda + 1)\,(v_1)\,\mathfrak{q}, \lambda v_1), \tag{14}$$

for all $\lambda, s_1 \in F$ and $v_1 \in V$.

Asymmetric product of F-braces - particular case

In our construction the condition (11), that guarantees the compatibility between sum and circle operation is quite difficult. We may require hypothesises that make easier condition (11).

Proposition

Let us note that if S and T are $\mathbb{F}$-vector spaces and seen as the zero $\mathbb{F}$-braces defining the operations $\circ$ such that $s_1 \circ s_2 = s_1 + s_2$ and $t_1 \circ t_2 = t_1 + t_2$, for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$ respectively.
In this case the equation (11) become:

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Example

Let p be an odd prime, $n, m \in \mathbb{N}$, $n \geq 3$, $q := p^m$ and $\mathbb{F}_q$ be the field of q elements. Let

- $\mathfrak{q} : \mathbb{F}_q^n \to \mathbb{F}_q$ be a non-degenerate quadratic form
- $A_1, \ldots, A_m \in O(V) = \{A \in GL(n, q) \mid \forall v \in \mathbb{F}_q^n \; v\mathfrak{q} = (vA)\mathfrak{q}\}$ of order p that commute by twos.
- X be the matrix associated respect a fixed basis to the bilinear form

$$(v, w)\mathfrak{b} = v\mathfrak{q} + w\mathfrak{q} - (v + w)\mathfrak{q} = -(v, w)\mathfrak{b}_\mathfrak{q}$$

  for all $v, w \in \mathbb{F}_q^n$
- $\alpha : (\mathbb{F}_q, +) = \bigoplus_{i=1}^{m} < \omega_i > \longrightarrow GL(n, \mathbb{F}_q)$ be the group homomorphism such that $\omega_i \alpha = A_i$.

Example

Let p be an odd prime, $n, m \in \mathbb{N}$, $n \geq 3$, $q := p^m$ and $\mathbb{F}_q$ be the field of q elements. Let

- $\mathfrak{q} : \mathbb{F}_q^n \to \mathbb{F}_q$ be a non-degenerate quadratic form
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Then $\mathfrak{b}$ and α satisfy hypothesis of the precedent result.

The adjoint group G of the asymmetric product $\mathbb{F}_q \ltimes_\circ \mathbb{F}_q^n$ is a regular subgroup of the affine group $AGL(n, p^m)$. Embedding $AGL(n, p^m)$ in $GL(n+1, p^m)$, the matrices associated to every elements

$\tau_{\left(1, \sum_{i=1}^{m} \mu_i \omega_i, v_2\right)} = \gamma_{\left(1, \sum_{i=1}^{m} \mu_i \omega_i, v_2\right)} t_{\left(1, \sum_{i=1}^{m} \mu_i \omega_i, v_2\right)}$ are

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$$G = \left\{ \begin{pmatrix} 1 & \sum_{i=1}^{m} \mu_i \omega_i + v_2 \mathfrak{q} & v_2 \\ 0 & 1 & 0 \\ 0 & \prod_{i=1}^{m} A_i^{\mu_i} X v_2^T & \prod_{i=1}^{m} A_i^{\mu_i} \end{pmatrix} \middle| \left(\sum_{i=1}^{m} \mu_i \omega_i, v_2 \right) \in \mathbb{F}_q \times \mathbb{F}_q^n \right\}. \qquad (18)$$

Now, if $m = 1$ and $n \geq 3$ then

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The subgroup G is isomorphic to the regular subgroup of $AGL(n,p)$ constructed by Hegedüs.

An analogue construction can be obtained in characteristic 2.

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