Regular subgroups of an affine group

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The affine group of a vector space

Let V be a vector space over a field F. The *affine group* $AGL(V)$ of V is the group generated by $GL(V)$ and $T(V)$, $AGL(V) := GL(V) \ltimes T(V)$, where $GL(V)$ is the *group of invertible linear maps* of V and $T(V)$ is the *translation group* of V.

A permutation group G over a set X is called *regular* if, for all $x, y \in X$, there exists a unique $\pi \in G$ such that $x\pi = y$.

Clearly $T(V)$ and its conjugated subgroups by an element of $GL(V)$ are abelian regular subgroups of $AGL(V)$.

Problem [M. W. Liebeck, C. E. Praeger, J. Saxl, 2009]

Find all regular subgroups of $AGL(V)$.

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An approach based on algebraic structure

In 2006, A. Caranti, F. Dalla Volta and M. Sala obtained a simple description of **all abelian regular subgroups** of the affine group $AGL(V)$ in terms of **radical commutative associative F-algebras** that have V as underlying vector space.

In 2009, F. Catino and R. Rizzo generalized this result obtaining a complete description of **all regular subgroups** of the affine group $AGL(V)$ in terms of **radical F-braces** that have V as underlying vector space.

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F-brace definition

Definition (F- brace)

Let V be a vector space over a field F and let $\cdot$ an operation on V. We call $V^\bullet := (V, +, \cdot)$ an *F-brace* if, for all $x, y, z \in V$ and for all $\lambda \in F$, the following conditions hold:

1. $(x + y) \cdot z = x \cdot z + y \cdot z$;
2. $x \cdot (y + z + y \cdot z) = x \cdot y + x \cdot z + (x \cdot y) \cdot z$
3. $\lambda(x \cdot y) = (\lambda x) \cdot y$.

Clearly every associative F-algebra is a F-brace.

On the other hand, every commutative F-brace is an associative F-algebra.

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Radical F-brace

Let V be an F-brace, as for associative F-algebras, we may introduce the *adjoint operation* on V setting

$$u \circ v := u + v + u \cdot v,$$

for all $u, v \in V$.

In general $(V, \circ)$ is a semigroup. If $(V, \circ)$ is a group, we say that the F-brace V is *radical*.

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Radical brace - characterization

If V is a radical F-brace and $\circ$ is the adjoint operation then $(V,+)$ is a vector space over F, $(V,\circ)$ is a group and the conditions

$$(v+w)\circ z + z = v\circ z + w\circ z \tag{1}$$

$$\lambda(v\circ w) = (\lambda v)\circ w + (\lambda - 1)w. \tag{2}$$

hold for all $v, w, z \in V$ and for all $\lambda \in F$. Conversely, if $(V,+)$ and $(V,\circ)$ are a vector space over F and a group respectively that satisfy equations (1) and (2), then posed $v \cdot w := v\circ w - v - w$ for all $v, w \in V$, we have that $(V,+,\cdot)$ is an radical F-brace.

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Link between radical F-braces and regular subgroups (I)

Theorem (Catino, Rizzo, 2009)

Let V be a vector space over a field F. Denote by $\mathcal{RB}$ the class of radical F-brace with underling vector space V and by $\mathcal{T}$ the set of all regular subgroups of the affine group $AGL(V)$.

1. *Let $V^\bullet \in \mathcal{RB}$. Then*
$$T(V^\bullet) = \{\tau_x | x \in V\},$$
where $\tau_x : V \to V, y \mapsto y \circ x$, is a regular subgroup of the affine group $AGL(V)$.
2. *The map*
$$f : \mathcal{RB} \to \mathcal{T}, \quad V^\bullet \mapsto T(V^\bullet)$$
is a bijection.

In this correspondence, isomorphism classes of F-brace correspond to conjugacy classes under the action of $GL(V)$ of regular subgroups of $AGL(V)$.

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Link between radical F-brace and regular subgroup (II)

Let F be a field, $n \in \mathbb{N}$ and $V := F^n$. By immersion of $AGL(n, F)$ in $GL(n+1, F)$ and by previous Theorem we have that if

$$T = \left\{ \begin{pmatrix} 1 & v \\ 0 & \gamma_v \end{pmatrix} \middle| v \in F^n \right\}$$

is a regular subgroup of $AGL(n, F)$, then there exists a unique radical F-brace $V^\bullet$ such that $T = T(V^\bullet)$.

Moreover the adjoint operation on $V^\bullet$ is

$$v \circ w = v\gamma_w + w = v\gamma_w t_w = v\tau_w,$$

for all $v, w \in V$.

That is, fixed a regular subgroup T of the affine group $AGL(n, F)$, there exists a unique radical F-brace on F^n such that its adjoint group is isomorphic to T.

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Link between radical F-brace and regular subgroup (II)

For example, let F be a field and let τ an endomorphism of the additive group of F. It is easy to see that the group

$$G = \left\{ \left(\begin{array}{ccc} 1 & x & y \\ 0 & 1 & x\tau \\ 0 & 0 & 1 \end{array} \right) \middle| \; x, y \in F \right\} \tag{3}$$

is a regular subgroup of the affine group $AGL(2, F)$

By previous result this, is in correspondence with the F-brace with underling vector space F^2 such that

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) \circ (x_2, y_2) = (x_1, y_1) \left(\begin{array}{cc} 1 & x_2\tau \\ 0 & 1 \end{array} \right) + (x_2, y_2) = (x_1 + x_2, x_1(x_2\tau) + y_1 + y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in F^2$.

Link between radical F-brace and regular subgroup (III)

Conversely, if $V^{\bullet}$ is a radical F-brace, set $\gamma_w : V \to V$, $v \mapsto v \circ w - w$, for all $w \in V$, and consider

$$T = \left\{ \begin{pmatrix} 1 & v \\ 0 & \gamma_v \end{pmatrix} \middle| v \in V \right\},$$

it is a regular subgroup of $AGL(n, F)$ and it is isomorphic to the adjoint group of the radical F-brace $V^{\bullet}$.

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Link between radical F-brace and regular subgroup (IV)

For example, if we consider the previous F-brace with underling vector space F^2 such that

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with τ an endomorphism of the additive group of F and $x_1, x_2, y_1, y_2 \in F$ we may compute the regular subgroup of $AGL(2, F)$ corresponding to this F-brace. In fact, let $\{(1, 0), (0, 1)\}$ be the canonical basis of F^2 Then, for all x, y

$$(1, 0)\gamma_{(x,y)} = (1, 0) \circ (x, y) - (x, y) = (1 + x, x\tau + y) - (x, y) = (1, x\tau)$$
$$(0, 1)\gamma_{(x,y)} = (0, 1) \circ (x, y) - (x, y) = (x, 1 + y) - (x, y) = (0, 1)$$

and so the regular subgroup is

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x\tau \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y \in F \right\},$$

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A cohomological approach

We introduce some cohomological tools in analogy with the method employed by W. A. de Graaf for the classification of nilpotent associative algebras of dimensions 2 and 3 over any field. In particular, we translate the concepts of "*2-cocycles*" and the "*Hochschild product*" from the context of associative F-algebras into that of F-braces.

In this way we obtain a description of all finite dimensional radical F-braces with non-trivial annihilator.

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The annihilator of an F-brace

We define the set of *right annihilator* of an F-brace V and that of *left annihilator* respectively as follows:

$$Ann_R(V) := \{x \mid x \in V,\ \forall\ v \in V,\ \forall\ \lambda \in F\ v \cdot (\lambda x) = 0\}$$

and

$$Ann_L(V) := \{x \mid x \in V,\ \forall\ v \in V,\ x \cdot v = 0\}.$$

These sets are subspaces of V. Note that the previous definitions of left and right annihilator cannot be symmetric, since in general, if $v, w \in V$ and $\lambda \in F$, then $v \cdot (\lambda w) \neq \lambda(v \cdot w)$.
The set $Ann(V) := Ann_L(V) \cap Ann_R(V)$ is called the *annihilator* of the F-brace V.

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2-cocycles of F-braces

Definition

Let A be an F-brace and V a vector space over a field F. A map $\theta : A \times A \to V$ with the properties:

1. $(\lambda a + \mu b, c)\theta = \lambda((a, c)\theta) + \mu((b, c)\theta)$,
2. $(a, b + c + b \cdot c)\theta = (a, b)\theta + (a, c)\theta + (a \cdot b, c)\theta$,

for all $a, b, c \in A$ and $\lambda, \mu \in F$, is called a *2-cocycle* of A with values in V.

Thus 2-cocycles of F-algebras [see for instance, R. S. Pierce, *Associative Algebras*] are particular cases of 2-cocycles of F-braces. But if we regard an F-algebra as an F-brace, then a 2-cocycle in the sense of the previous definition is not necessarily a 2-cocycle in the usual sense.

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Thus 2-cocycles of F-algebras [see for instance, R. S. Pierce, *Associative Algebras*] are particular cases of 2-cocycles of F-braces. But if we regard an F-algebra as an F-brace, then a 2-cocycle in the sense of the previous definition is not necessarily a 2-cocycle in the usual sense.

2-cocycles of F-braces

Definition

Let A be an F-brace and V a vector space over a field F. A map $\theta : A \times A \to V$ with the properties:

1. $(\lambda a + \mu b, c)\theta = \lambda((a, c)\theta) + \mu((b, c)\theta)$,
2. $(a, b + c + b \cdot c)\theta = (a, b)\theta + (a, c)\theta + (a \cdot b, c)\theta$,

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Thus 2-cocycles of F-algebras [see for instance, R. S. Pierce, *Associative Algebras*] are particular cases of 2-cocycles of F-braces. But if we regard an F-algebra as an F-brace, then a 2-cocycle in the sense of the previous definition is not necessarily a 2-cocycle in the usual sense.

Example

Let N be the zero F-algebra of dimension n over a field F and τ an endomorphism of the additive group of F. Then the map

$$\theta : N \times N \longrightarrow F, \left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i\right) \longmapsto \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\tau \qquad (4)$$

is a 2-cocycle of the F-brace N but, in general, not of the F-algebra N.

In particular, θ is a 2-cocycle of the n-dimensional zero F-algebra if and only if τ is linear.

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Hochschild product

Definition

Let A be an F-brace, V an F-vector space, $\theta : A \times A \to V$ a 2-cocycle. Put $A_\theta := A \oplus V$. For all $a, b \in A$ and $v, w \in V$ we define

$$(a + v) \cdot (b + w) := a \cdot b + (a, b)\theta. \quad (5)$$

The F-brace A_θ is called a *Hochschild product* of A by V.

What may we say of A_θ if A is a radical F-brace?

If A is a radical F-brace and θ is a 2-cocycle of A with values in an F-vector space V. Then A_θ is radical.

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Description of F-braces with non-trivial annihilator (I)

Theorem (F. Catino, I. C., P. Stefanelli, *Bull. Aust. Math. Soc.*, 2015)

Let B be a radical F-brace such that $Ann(B) \neq \{0\}$. Then there exist an F-brace A, an F-vector space V and a 2-cocycle $\theta : A \times A \to V$ such that B is isomorphic to A_θ.

We consider $A := B/Ann(B)$, $V := Ann(B)$.
If $\pi : B \to A$ be the projection map and we choose a linear map $\sigma : A \to B$ such that $(x\sigma)\pi = x$, for all $x \in A$, then we obtain a function θ from $A \times A$ into V by defining

$$(x, y)\theta := x\sigma \cdot y\sigma - (x \cdot y)\sigma. \tag{6}$$

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The intersection with the translation group (I)

The regular subgroups of an affine group obtained by the Hochschild product of radical F-braces have non-trivial intersection with the translation group.

Proposition (F. Catino, R. Rizzo, 2009)

Let $V^\bullet$ be a radical F-brace with underlying vector space V over a field F. Let $T(V^\bullet) = \{(\gamma_a, a) \mid a \in V\}$ and $T(V)$ be the translation group. Then

$$T(V) \cap T(V^\bullet) = \{(\gamma_a, a) \mid a \in \mathrm{Soc}(V^\bullet)\}, \qquad (7)$$

where $\mathrm{Soc}(V^\bullet) := \{x \mid x \in V, \forall\, v \in V \; v \cdot x = 0\}$ is the socle of V.

Let us note that $Ann_R(V) \subseteq Soc(V)$.

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The intersection with the translation group (II)

The Hochschild Product of radical F-braces is not exhaustive because there are examples of subgroups of affine group with trivial intersection with the translations as the following:

Theorem (P. Hegedűs, 2000)

Let p be a prime. If $p = 2$ then assume $n = 3$, or $n \geq 5$. If p is odd then assume $n \geq 4$. Then the affine group $AGL(n, p)$ has a regular subgroup which contains no translations other than the identity.

Hegedűs' subgroups (I)

Let p be a prime and $n \in \mathbb{N}$, if p is odd, let $n \geq 4$ otherwise if $p = 2$, let $n \geq 3$ odd. Over the field $\mathbb{F}_p$ of p elements. Consider the immersion of $AGL(n, p)$ into $GL(n+1, p)$ and let

- $\mathfrak{q} : \mathbb{F}_p^{n-1} \to \mathbb{F}_p$ be a non-degenerate quadratic form;
- $\mathfrak{b} : \mathbb{F}_p^{n-1} \times \mathbb{F}_p^{n-1} \to \mathbb{F}_p$ be the symmetric bilinear form associated to $\mathfrak{q}$;
- X the matrix associated to $\mathfrak{b}$ respect to a fixed basis $((v+w)\mathfrak{q} = v\mathfrak{q} + w\mathfrak{q} + vXw^T$ for all $v, w \in \mathbb{F}_p^{n-1})$;
- A the orthogonal non-singular matrix $(n-1) \times (n-1)$ of order p $(XA^T = A^{-1}X)$ and such that $v\mathfrak{q} = (vA)\mathfrak{q}$ for all $v \in \mathbb{F}_p^{n-1}$.

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Hegedűs' subgroups (II)

Then the set

$$T := \left\{ \begin{pmatrix} 1 & m & v \\ 0 & 1 & 0 \\ 0 & A^m X w^T & A^m \end{pmatrix} \middle| \, m \in \mathbb{F}_p, v \in \mathbb{F}_p^{n-1} \right\}.$$

is a regular subgroup of the affine group $AGL(n, p)$ that has trivial intersection with the translation group.

In 2016, F. Catino, I. C., P. Stefanelli, *J. Algebra* 455 (2016), 164–182, introduced a construction of radical F-braces that has as **very particular case** a generalization of the Hegedűs' subgroup.

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The Asymmetric Product of zero F-braces (I)

Let

- F be a field;
- V be a n-dimensional vector space over F (reviewed as **zero F-brace**);
- $\mathfrak{q} : V \to F$ be a quadratic form and $\mathfrak{b} : V \times V \to F$ the polar form of $\mathfrak{q}$;
- $\alpha : F \to \mathrm{Aut}(V)$ be a group homomorphism from $(F, +)$ to the automorphism group of the F-brace V.

If

$$(v^s)\mathfrak{q} = (v)\mathfrak{q} \tag{8}$$

holds for all $v \in V$ and $s \in F$, then we may define over $F \times V$ a structure of radical F-brace.

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The Asymmetric Product of zero F-braces (II)

Set the sum, the multiplication over $F \times V$ and the scalar multiplication

$$(s_1, v_1) + (s_2, v_2) = (s_1 + s_2 + (v_1, v_2)\mathfrak{b}, v_1 + v_2) \tag{9}$$

$$(s_1, v_1) \circ (s_2, v_2) = (s_1 \circ s_2, v_1^{s_2} + v_2) \tag{10}$$

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In this cases we may check the intersection of the regular subgroup associated to the radical F-brace with the translation group through properties of $\mathfrak{b}$ and α.

The regular subgroup of $AGL(n+1, F)$ associated with $F \ltimes_\circ V$ intersects trivially the translation group if and only if the symmetric bilinear form $\mathfrak{b} : V \times V \to F$ (the quadratic form $\mathfrak{q} : V \to F$, respectively) is non-degenerate and the action $\alpha : F \to \mathrm{Aut}(V)$ is faithful.

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In this cases we may check the intersection of the regular subgroup associated to the radical F-brace with the translation group through properties of $\mathfrak{b}$ and α.

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The Asymmetric Product of zero F-braces (II)

Set the sum, the multiplication over $F \times V$ and the scalar multiplication

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Generalization of Hegedűs' subgroups (I)

Let p be a prime and $n \geq 4$ (if p is odd) or $n \geq 4$ even (if $p = 2$). Set $V = \mathbb{F}_{p^m}^n$ and consider

1. $\mathfrak{q} : V \to \mathbb{F}_{p^m}$ an isotropic quadratic form such that its polar form $\mathfrak{b}$ is non-degenerate.
2. $A_1, \dots, A_m \in O(V, \mathfrak{q}) = \{A \in GL(n, p^m) \mid \forall v \in \mathbb{F}_{p^m}^n \ \ v\mathfrak{q} = (vA)\mathfrak{q}\}$ distinct of order p that commutes two by two.
3. $\alpha : \mathbb{F}_{p^m} = \bigoplus_{i=1}^{m} \langle \omega_i \rangle \longrightarrow GL\left(n, \mathbb{F}_{p^m}\right)$ a group homomorphism such that $\omega_i \alpha = A_i$ (α is clearly injective)

Then $\mathfrak{b}$ and α are compatible and we may consider the radical brace over $\mathbb{F}_{p^m}$ asymmetric product $\mathbb{F}_{p^m} \ltimes_{\circ} V$.

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Generalization of Hegedűs' subgroups (II)

Let X the matrix associated to $-\mathfrak{b}$ respect a fixed basis, then the regular subgroup of the affine group associated to this $\mathbb{F}_{p^m}$-brace is given by

$$G = \left\{ \begin{pmatrix} 1 & \sum_{i=1}^{m} \mu_i \omega_i + (v_2)\,\mathfrak{q} & v_2 \\ 0 & 1 & 0 \\ 0 & \left(\prod_{i=1}^{m} A_i^{\mu_i}\right) X v_2^T & \prod_{i=1}^{m} A_i^{\mu_i} \end{pmatrix} \middle| \left(\sum_{i=1}^{m} \mu_i \omega_i, v_2 \right) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}^n \right\}.$$

If $m = 1$ then G becomes

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Thank you for your attention.