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Cabling for non-involutive solutions

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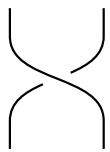
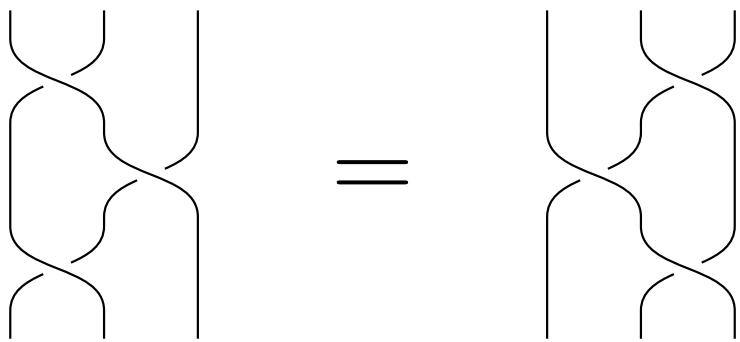
The interplay between skew braces and Hopf–Galois theory

Solutions of the Yang-Baxter equation

A set-theoretic solution (to the YBE) is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a bijective map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r). \qquad (*)$$

Write $r =$  . Then $(*)$ becomes



Set-theoretic solutions to the Yang-Baxter equation

Let (X, r) be a set-theoretic solution to the YBE. Write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

- (X, r) is involutive if $r^2 = \mathrm{id}$.
- (X, r) is finite if X is finite.
- (X, r) is non-degenerate if λ_x and ρ_x are bijective for all $x \in X$.

Examples

X a set.

- $r(x, y) = (y, x)$ is an involutive non-degenerate solution.
- f, g permutaion of X. Then $r(x, y) = (f(y), g(x))$ is a solution if and only if $fg = gf$.

  Morever, (X, r) is involutive if and only if $g = f^{-1}$.

  (X, r) is called a permutational solution or a Lyubashenko's solution.

G a group.

- $r(x, y) = (y, y^{-1}xy)$ is a bijective non-degenerate solution.

Convention

From now on

solution $=$ finite bijective non-degenerate set-theoretic solution to the YBE.

Indecomposable solutions

A solution (X, r) is decomposable if there exists a partition of X (i.e. $\emptyset \neq Y, Z \subseteq X$ such that $X = Y \cup Z$ and $Y \cap Z = \emptyset$) s.t.

$$r(Y \times Y) \subseteq Y \times Y \quad \text{and} \quad r(Z \times Z) \subseteq Z \times Z.$$

Otherwise, the solution is said to be indecomposable.

Indecomposable solutions

Fact. A solution (X, r) is indecomposable if and only if the group

$$\mathrm{gr}(\lambda_x, \rho_y : x, y \in X)$$

acts transitively on X.

Indecomposable solutions

Example

- X a set with n elements.
- f a cycle of length n.
- Then $r : X \times X \to X \times X, (x, y) \mapsto (f(y), x)$ is an **indecomposable solution**.

Problem. Construct indecomposable solutions.

Involutive indecomposable solutions

Facts. Let (X, r) be an **involutive** solution. Then

- $\rho_y(x) = \lambda^{-1}_{\lambda_x(y)}(x)$, for all $x, y \in X$.
- (X, r) is indecomposable if and only if $\mathrm{gr}(\lambda_x : x \in X)$ is transitive on X.

The diagonal map

Let (X, r) be a **involutive** solution. The map $T : X \to X$ defined by

$$T(x) = \lambda_x^{-1}(x).$$

is bijective and it is called the diagonal map.

Important. The cycle decomposition of T is an invariant for solutions and gives information about decomposability.

Square-free solutions

A solution (X, r) is square-free if $r(x, x) = (x, x)$ (i.e., $T = \mathrm{id}$).

Theorem (Rump, conjecture by Gateva-Ivanova)**.** If (X, r) is a square-free **involutive** solution, then (X, r) is decomposable.

Problem. What can we say about the cycle decomposition of T for (in)decomposable solutions?

Some results

Let (X, r) be a solution and assume $|X| = n$.

(Ramírez & Vendramin)

- If T is a n-cycle, then (X, r) is indecomposable.
- If T is a $(n-1)$-cycle, then (X, r) is decomposable.
- If T is a $(n-2)$-cycle, n odd, then (X, r) is decomposable.
- If T is a $(n-3)$-cycle, $\gcd(n, 3) = 1$ odd, then (X, r) is decomposable.

(Camp-Mora & Sastriques)

- If $\gcd(|T|, n) = 1$, then (X, r) is decomposable.

Skew braces

A skew brace is a triple $(B, +, \circ)$ such that $(B, +)$ and $(B, \circ)$ are (not necessarily abelian) groups and the following holds

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

for all $a, b, c \in B$.

- $(B, +)$ is the additive structure of $(B, +, \circ)$.
- $(B, \circ)$ is the multiplicative structure of $(B, +, \circ)$.

The structure group

Let (X, r) be a solution. The group defined by

$$G(X, r) = \mathrm{gr}(X \mid x \circ y = \lambda_x(y) \circ \rho_y(x))$$

is structure group of (X, r).

If (X, r) is an **involutive**, then $G(X, r)$ has a structure of skew brace with additive structure isomorphic to $\mathbb{Z}^{|X|}$.

Facts.

- If B is a skew brace, then $r_B(a,b) = (-a + a \circ b, (-a + a \circ b)' \circ a \circ b)$ is a solution.
  If, in addition, $(B, +)$ is abelian then r_B is involutive.
- If (X, r) is an **involutive** solution then (X, r) extends to $(G(X, r), r_{G(X,r)})$.
- If (X, r) is an **involutive** solution then $\iota : X \to G(X, r)$, $x \to x$ is injective.

Idea: cabling

Lebed, Ramírez & Vendramin

Let (X, r) be an involutive solution. For $k \geq 1$, the map $\iota^{(k)} : X \to G(X, r)$, $x \mapsto kx$ is injective.

$$(X, r) \xrightarrow{\text{extend}} (G(X, r), r_{G(x,r)}) \qquad \text{pull-back using } \iota^{(k)}$$

$$\downarrow r^{(k)}$$

k-cabled solution

Theorem (Lebed, Ramírez & Vendramin)**.** Let (X, r) be an involutive solution.

- The diagonal map of $r^{(k)}$ is T^k.
- If (X, r) **indecomposable** and $\gcd(|X|, k) = 1$, then $r^{(k)}$ is **indecomposable**.

Taking $k = |T|$ Camp-Mora & Sastriques theorem reduces to Rump's theorem.

Question. What about cabling for non-involutive solutions?

Main issues (1)

Let (X, r) be a solution. One of the main issues is that $\iota : X \to G(X, r)$, $x \mapsto x$ **is not an injective map**.

Example.

- $X = \{1, 2, 3, 4\}$.
- $f = (1\ 2)$ and $g = (3\ 4)$.
- $r(x, y) = (f(y), g(x))$ is a solution.

- (X, r) is not injective.

  Indeed, in $G(X, r)$ we have $1 = 2$ and $3 = 4$.

The injectivization

Let (X, r) be a e solution and let $\iota : X \to G(X, r)$ $x \mapsto x$. Then

$$\mathrm{Inj}(X, r) = (\iota(X), r_{G(X,r)|_{\iota(X)\times\iota(X)}})$$

is a solution called the injectivization of (X, r).

Fact. It holds that

$$G(X, r) \cong G(\iota(X), r_{G(X,r)|_{\iota(X)\times\iota(X)}}).$$

Injective solutions

A solution (X, r) is injective if the map $\iota : X \to G(X, r)$ is injective.

Examples.

- (X, r) a solution $\mathrm{Inj}(X, r)$ is an injective solution.
- Solutions associated to skew braces are injective.
- Irretractable solutions are injective.

We can focus on injective solutions

Theorem (IC & Van Antwerpen)**.** Let (X, r) be a solution. Then

$$(X, r) \text{ is } \textbf{decomposable} \iff \mathrm{Inj}(X, r) \text{ is } \textbf{decomposable}.$$

Hence, we can focus simply on **injective** solutions.

Main issues (2)

Recall that in the definition of the k-cabled solution, it was crucial that the map $\iota^{(k)} : X \to G(X, r)$, $x \mapsto kx$ **is injective**.
However, this **fails** even for injective solutions.

Example.

- $X = \{x_1, x_2, x_3\}$.
- $\sigma_1 = (2\ 3)$, $\sigma_2 = (1\ 3)$ and $\sigma_3 = (1\ 2)$.

The solution

$$r(x_j, x_k) = (x_k, x_{\sigma_k(j)})$$

is injective and indecomposable.

But in $G(X, r)$ one has that $2x_1 = 2x_2 = 2x_3$.

The structure monoid

Let (X, r) be a solution. The structure monoid is the monoid

$$M(X, r) = \langle X \mid x \circ y = \lambda_x(y) \circ \rho_y(x) \rangle.$$

Facts (Gateva-Ivanova & Majid, Lebed &Vendramin)**.**

- If (X, r) is a solution then (X, r) extends in a unique way a solution r_M on $M(X, r)$ such that

$$r_{M(X,r)}(\iota \times \iota) = (\iota \times \iota)r$$

  where $\iota : X \to G(X, r)$ is the canonical map.

- $M(X, r) \overset{\text{regular}}{\hookrightarrow} A(X, r) \rtimes \operatorname{Sym} X$, where $A(X, r) = \langle X \mid x + z = z + \sigma_z(x) \rangle$ is the structure monoid associated to the derived solution.

k-cabled solutions

Prop (IC, Van Antwerpen)**.** Let (X, r) be an **injective** solution. Then $kX = \{(kx, \lambda_{kx})\} \subseteq M(X, r)$ defines a subsolution (kX, r_k) of $(M(X, r), r_M)$.

Definition. Let (X, r) be an **injective** solution and let $r^{(k)} = (\varphi_k^{-1} \times \varphi_k^{-1}) r_k (\varphi_k \times \varphi)$ where $\varphi_k : X \to kX, x \mapsto kx$. Then $(X, r^{(k)})$ is the k-cabled solution.

Prop. Let (X, r) be an injective solution.

- If k is an integer, then $(X, r^{(k)})$ is injective.
- If k, k' are integers, then $(X, (r^{(k)})^{(k')}) = (X, r^{(kk')})$.

Theorem (IC, Van Antwerpen)**.** Let (X, r) be an injective solution.

- The diagonal map of $r^{(k)}$ is T^k.
- If (X, r) **indecomposable** and $\gcd(|X|, k) = 1$, then $r^{(k)}$ is **indecomposable**.

Decomposability results

Theorem (Darné). Let $(X, \triangleleft)$ be a rack with $|X| > 1$ such that $x \triangleleft x = x$ (i.e. $(X, \triangleleft)$ is a quandle), and let $(X, r_\triangleleft)$ the solutions associated to $(X, \triangleleft)$. If the structure group $G(X, r_\triangleleft)$ is **nilpotent** and not isomorphic to $\mathbb{Z}$, then $(X, r_\triangleleft)$ is **decomposable**.

We obtained a completely group-theoretical proof of this result.

Corollary. Let $(X, \triangleleft)$ be a rack and let $(X, r_\triangleleft)$ the solutions associated to $(X, \triangleleft)$. If the structure group $G(X, r_\triangleleft)$ is **nilpotent** and not isomorphic to $\mathbb{Z}$, then $(X, r_\triangleleft)$ is **decomposable**.

Square-free solutions

Theorem (IC, Van Antwerpen) **.** Let (X, r) be a solution and (X, s) its derived solution. If (X, r) is square-free and $A_g(X, r) = G(X, s)$ is **nilpotent**, then (X, r) is decomposable.

Thank you!!!