Radical braces and the Yang-Baxter equation

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Yang-Baxter operators

The Yang-Baxter equation is a basic equation of the statistical mechanics that arose from Yang's work in 1967 and Baxter's one in 1972.

Let V be a vector space over a field F. If $R : V \otimes V \to V \otimes V$ is a linear map, set

$$R_{12} = R \otimes id_V, \quad R_{23} = id_V \otimes R, \quad R_{13} = (\sigma_{V,V} \otimes id_V)(id_V \otimes \sigma_{V,V}) \tag{1}$$

where $\sigma : V \otimes V \to V \otimes V$ is the twist operator given by $\sigma(v \otimes w) = w \otimes v$.

A linear map $R : V \otimes V \to V \otimes V$ is called a Yang-Baxter operator if R is a solution of the quantum Yang-Baxter equation, that is,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2}$$

holds in the monoid of the linear maps of $V \otimes V \otimes V$ to itself.

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where $\gamma : V \otimes V \to V \otimes V$ is the twist operator given by $\gamma(v \otimes w) = w \otimes v$.

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where $\tau : V \otimes V \to V \otimes V$ is the *twist operator* given by $\tau(u \otimes v) = v \otimes u$.

A linear map $R : V \otimes V \longrightarrow V \otimes V$ is called a **Yang-Baxter operator** if it is a solution of the quantum Yang-Baxter equation, i.e.,

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Set-theoretic solutions of the quantum Yang-Baxter equation

A simplification

In 1992 Drinfeld suggested to study a simplified case: fixed a basis B on a vector space V we may find all solutions R induced by a linear extension of the map $\mathcal{R} : B \times B \to B \times B$.

In this case, the following equivalence is useful:

If B is a non-empty set, denoted by $\tau : B \times B \to B \times B$ the *twist map*, i.e., $\tau(x, y) = (y, x)$, we have that $\mathcal{R} : B \times B \to B \times B$ is a Yang-Baxter map if and only if the function $r := \tau\mathcal{R} : B \times B \to B \times B$ satisfies the **braided equation**

$$r_1 r_2 r_1 = r_2 r_1 r_2, \tag{3}$$

where $r_1 := r \times id_B$ and $r_2 := id_B \times r$.

Usually, r or $\mathcal{R}$ is called a **set-theoretic solution of the quantum Yang-Baxter equation**.

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Simple examples of solutions of the Yang-Baxter equation (I)

Let B be a set. The following are solutions of the Yang-Baxter equation.

1. The twist map, i.e., $r : B \times B \to B \times B$ such that $r(x, y) = (y, x)$;
2. The map $r : B \times B \to B \times B$ such that

$$r(x, y) = (f(y), g(x))$$

where f, g are functions from B to itself such that $fg = gf$. Introduced by V.V. Lyubashenko.

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Simple examples of solutions of the Yang-Baxter equation (II)

3. The map $r : B \times B \to B \times B$ such that

$$r(x, y) = (y, y * x),$$

where $*$ is an operation on B such that

$$x * (y * z) = (x * y) * (x * z), \tag{4}$$

for all $x, y, z \in B$. Introduced by B.B. Venkov. For instance, if B is a group setting $y * x := y^{-1}xy$ (for all $x, y \in B$) then $*$ satisfies relation (4).

4. Let B be a group and $f : B \to B$ a group endomorphism such that $f^2 = f$. The the map $r : B \times B \to B$ such that

$$r(x, y) = (f(y), f(y)^{-1}xy)$$

for all $x, y \in B$, is a solution of the Yang-Baxter equation. Note that in $f = \mathrm{id}_B$ we obtain again the Venkov's solution in the particular case in which $*$ is the operation of conjugation.

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Non-degenerate involutive set-theoretic solutions

Determining all solutions of the Yang-Baxter equation is a very difficult task.
Although we may find several works about this topic, it is still an open problem.

In 1999 Etingof, Schedler and Soloviev laid the groundwork for the study of a particular class of these solutions, the *non-degenerate involutive* ones.

Definition

Let B be a set, $\lambda_x, \rho_y : B \times B \to B \times B$ and $r : B \times B \to B \times B$ defined by $r(x, y) = (\lambda_x(y), \rho_y(x))$ that satisfies the braided equation. Then r is said to be **non-degenerate involutive set-theoretic solution** if

1. $\forall\, x \in B \quad \lambda_x, \rho_x \in \mathrm{Sym}_B$;
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Examples (I)

1. Clearly, if B is a set, then the twist map $r : B \times B \to B \times B$ such that $r(x, y) = (y, x)$ is non-degenerate involutive set-theoretic solution of the Yang-Baxter equation;

2. The Lyubashenko's solution, i.e.,

$$r : X \times B \longrightarrow B \times B,\ (x, y) \longmapsto (f(y), g(x))$$

is non-degenerate involutive set-theoretic solution of the Yang-Baxter equation if and only if f, g are bijective and $g = f^{-1}$;

3. Let B be group and f be a group endomorphism such that $f^2 = f$. Then the solution $r : B \times B \to B$ such that $r(x, y) = (f(y), f(y)^{-1}xy)$ is a non-degenerate set-theoretic solution of the Yang-Baxter equation if and only if $f = \mathrm{id}_B$. Moreover r is involutive if and only if B is abelian.

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Examples (II)

4. Let R be a radical ring. Then $(R, \circ)$ is a group where $\circ$ is the classical *adjoint operation*

$$\forall\, a, b \in R \quad a \circ b = a + b + ab.$$

$(R, \circ)$ is called the adjoint group. For all $b \in R$, let

$$\rho_b : R \longrightarrow R,\ a \longmapsto ab + a \tag{5}$$

Then the functions ρ_b are bijective. Set, for all $a \in R$,

$$\lambda_a : R \longrightarrow R,\ b \longmapsto \rho^{-1}_{\rho_b(a)}(b) \tag{6}$$

we have that the function

$$r : R \times R \to R \times R,\ (a, b) \longmapsto (\lambda_a(b), \rho_b(a)) \tag{7}$$

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4. Let R be a radical ring. Then $(R, \circ)$ is a group where $\circ$ is the classical *adjoint operation*

$$\forall\, a, b \in R \quad a \circ b = a + b + ab.$$

$(R, \circ)$ is called the adjoint group. For all $b \in R$, let

$$\rho_b : R \longrightarrow R,\ a \longmapsto ab + a \tag{5}$$

Then the functions ρ_b are bijective. Set, for all $a \in R$,

$$\lambda_a : R \longrightarrow R,\ b \longmapsto \rho^{-1}_{\rho_b(a)}(b) \tag{6}$$

we have that the function

$$r : R \times R \to R \times R,\ (a, b) \longmapsto (\lambda_a(b), \rho_b(a)) \tag{7}$$

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Braces

Definition

An abelian group B with a product $\cdot$ is said to be a **(right) brace** if the conditions

1. $(a+b)c = ac + bc$;
2. $a(b+c+bc) = ab + ac + (ab)c$,

hold for all $a, b, c \in B$.

Clearly, any associative ring is a brace and any commutative brace is an associative ring.

For instance, an abelian group B with the product defined by

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is a brace and if $B \neq 0$ it is not a ring. Moreover, if B is not an elementary 2-group then the multiplication is not associative.

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Radical braces

As for an associative ring if B is a brace we may introduce the **adjoint operation**:

$$\forall\, a, b \in B \quad a \circ b := a + b + ab.$$

Then the structure $(B, \circ)$ is a semigroup.

We say that a brace B is **radical** if $(B, \circ)$ is a group. In this case we call the adjoint group $(B, \circ)$ the **multiplicative group** of the brace B.

For instance, the additive group of the integers $(\mathbb{Z}, +)$ with the multiplication given by

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Radical braces

As reformulated by Cedó, Jespers and Okniński in 2014 the following definition is equivalent the original one of radical brace introduced by Rump.

Definition

A **radical (right) brace** is a set B with two operations $+$ and $\circ$ such that $(B,+)$ is an abelian group, $(B,\circ)$ is a group and

$$(a+b)\circ c + c = a\circ c + b\circ c \tag{9}$$

holds for all $a,b,c \in B$.

This reformulation of the definition of a radical brace leads to a clear connection between radical braces and solutions of the Yang-Baxter equation.

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1. The easiest example of brace is an abelian group $(G, +)$ with multiplication given by $a \circ b = a + b$, for all $a, b \in G$. This one is called **zero brace**.
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Let $(B, +, \circ)$ be a radical brace. Then, $(B, +, \cdot)$ is a ring, where $a \cdot b = a \circ b - a - b$, for all $a, b \in B$, if and only if

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Proposition

Let $(B, +, \circ)$ be a radical brace. Then, the function $r : B \times B \longmapsto B \times B$ given by

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A characterization of solutions of the Yang-Baxter equation (I)

The following result is very useful to prove that a map is a solution of the Yang-Baxter equation.

Theorem (Lu, Yan, Zhu, 2000)

Let G be a group. If a right action of G on itself and a left action of G on itself satisfy the compatibility condition

$$\forall\, x, y \in G \quad xy = ({}^{x}y)(x^{y}), \tag{12}$$

then the function $r : G \times G \to G \times G$, given by

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be the solution of Yang-Baxter equation associated to the brace B. Then, setting

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for all $a, b \in B$,we have that $\lambda(a) = \lambda_a$ is a left action and $\rho(b) = \rho_b$ is a right action of the multiplicative group $(B, \circ)$ on itself that satisfy the compatibility condition in Lu, Yan, Zhu's theorem.

Moreover,

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Examples

1. If $(B, +, \circ)$ is the zero brace, i.e. $a \circ b = a + b$, for all $a, b \in B$, then the solution associated to B is twist map $r : B \times B \to B \times B,\ (x, y) \mapsto (y, x)$.

2. If R is a radical ring, then the solution associated to R as radical brace, i.e., $r : R \times R \to R \times R$ such that $r(a, b)(a \circ (a + b^-)^-, (a + b^-) \circ b)$ coincides with that viewed before. In fact,

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3. If B is the brace with additive group $(\mathbb{Z}, +)$ with multiplication given by $x \circ y := (-1)^y + x$, for all $x, y \in \mathbb{Z}$, then the solution associated to B is the function $r : B \times B \to B \times B$ defined by

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The semidirect product of radical braces [Rump, 2008]

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For example, let A and B be radical braces and $\alpha : B \to \mathrm{Aut}(A)$ is a homomorphism from the multiplicative group of B into the group of automorphisms of the radical brace A. Then the sum and the multiplication over $A \times B$ given by

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$$(a_1, b_1) + (a_2, b_2) := (a_1 + a_2, b_1 + b_2) \tag{15}$$

$$(a_1, b_1) \circ (a_2, b_2) := (a_1 \circ a_2, b_1^{a_2} \circ b_2)\,, \tag{16}$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, define a structure of radical brace on $A \times B$, called **semidirect product** of A and B.

From radical rings we construct radical braces

As we expect, the semidirect product of two radical rings does not determine a structure of radical ring. Hence, we may easily construct radical braces through radical rings.

For instance,

- let $A = \mathbb{F}_2$ be the zero radical brace, i.e., such that $a_1 \circ a_2 = a_1 + a_2$, for all $a_1, a_2 \in A$;
- let $B = \mathbb{F}_3$ be the zero radical brace, i.e., such that $b_1 \circ b_2 = b_1 + b_2$, for all $b_1, b_2 \in B$;
- let $\alpha : A \to \mathrm{Aut}(B)$ be the action of the multiplicative group of A to the radical brace B given by $b^0 = b$, $b^1 = -b$, for all $b \in B$.

Then,

$$(0,1)\circ((1,1)+(0,1))+(0,1) = (0,1) \quad \text{but} \quad (0,1)\circ(1,1)+(0,1)\circ(0,1) = (1,2).$$

Remember that a radical brace is a radical ring if and only if

$$c \circ (a+b) + c = c \circ a + c \circ b \qquad \forall a, b, c \in B.$$

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- Let $\alpha : A \to \mathrm{Aut}(B)$ be the action of the group of $(A, \circ)$ to the radical brace B given by $b^0 = b$, $b^1 = -b$, for all $b \in B$.

Explicitly the adjoint operation obtained by the semidirect product $A \ltimes B$ is

$$(a_1, b_1) \circ (a_2, b_2) = \left(a_1 + a_2, (-1)^{a_2 + b_2} b_1 + b_2\right) \tag{17}$$

Then, the associate solution is

$$r\left((a_1, b_1), (a_2, b_2)\right) = \left(\left(a_2, (-1)^{a_1 + b_1} b_2\right), \left(a_1, (-1)^{a_2 + b_2} b_1\right)\right) \tag{18}$$

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The asymmetric product of radical braces

In 2016, F. Catino, I. C., P.Stefanelli, *J. Algebra* 455, 164–182, introduced the *asymmetric product of radical braces*, a new construction of radical braces which extends the semidirect product, that allows us to obtain new solutions of the Yang-Baxter equation.

In particular, this construction involves symmetric cocycles of abelian groups. We recall that, if A and B are abelian groups, then a function $\mathfrak{f} : B \times B \to A$ is a **symmetric cocycle** on B with values in A if $\mathfrak{f}(b_1, b_2) = \mathfrak{f}(b_2, b_1)$ and,

$$\mathfrak{f}(0,0) = 0, \tag{19}$$

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Main Theorem

Theorem (F. Catino, I. C., P.Stefanelli, *Journal of Algebra*, 2016)

Let A and B be radical braces and consider an action of the multiplicative group $(A, \circ)$ on the brace B. Let $\mathfrak{f} : B \times B \to A$ be a symmetric cocycle on $(B, +)$ with values in $(A, +)$ such that

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for all $a \in A$ and $b_1, b_2, b_3 \in B$.
Then, the addition and the multiplication over $A \times B$ given by

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Semidirect product as Asymmetric product

Let A and B be radical braces and consider an action of the multiplicative group $(A, \circ)$ on the brace B. Then we may note that $\mathfrak{f} : B \times B \to A$ such that

$$\mathfrak{f}(b_1, b_2) = 0_A$$

is a symmetric cocycle on $(A, +)$ with values in $(B, +)$.

Moreover

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In this way we may view the semidirect product of radical braces as particular case of the asymmetric product of radical braces.

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- Let $A = \mathbb{F}_2$ be the zero radical brace, i.e., such that $a_1 \circ a_2 = a_1 + a_2$, for all $a_1, a_2 \in A$;
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- Let $\mathfrak{f} : B \times B \to A$ be the symmetric cocycle on $(A, +)$ with values in $(B, +)$ defined by

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Example (II)

Then α and $\mathfrak{f}$ satisfy the condition (21). Explicitly the addition and the multiplication of asymmetric product $A \ltimes_{\circ} B$ are respectively

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2 + b_1 b_2, b_1 + b_2) \tag{24}$$

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The solution of the Yang-Baxter equation associated to the asymmetric product $A \ltimes_{\circ} B$ is the map $r : (A \times B) \times (A \times B) \to (A \times B) \times (A \times B)$ such that $r\left((a_1, b_1), (a_2, b_2)\right) = \left(\lambda_{(a_1, b_1)}(a_2, b_2), \rho_{(a_2, b_2)}(a_1, b_1)\right)$ where

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As already noted the semidirect product is a particular asymmetric product in which the symmetric cocycle is trivial. In this example if we consider the trivial symmetric cocycle we have again the example of the semidirect product of $\mathbb{F}_2$ (as zero brace) and the brace with additive group $(\mathbb{Z}, +)$ an multiplication given by $b_1 \circ b_2 = (-1)^{b_2} + b_2$. The action is the same.

Then we may compare the solutions of Yang-Baxter equation associate to the semidirect product and to the asymmetric product.

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Non-degenerate solution not necessary involutive

In 2015, L. Guarnieri, L. Vendramin introduced a more general algebraic structure the *skew brace* that generalizes radical brace. From this structure it is possible to construct new solutions of Yang-Baxter equation that are non-degenerate but not necessarily involutive.

Definition (L. Guarnieri, L. Vendramin, Accepted in *Math. Comp.*, 2015)

A **skew (right) brace** is a set B with two operations $+$ and $\circ$ such that $(B,+)$ is an group, $(B,\circ)$ is a group and

$$(a+b)\circ c = a\circ c - c + b\circ c \tag{30}$$

holds for all $a,b,c \in B$.

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Examples

1. Let $(B, +, \circ)$ be a radical brace then obviously B is a skew brace.
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Let $(B, +, \circ)$ be a skew brace. Then $r : B \times B \longmapsto B \times B$ given by

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In fact, as for braces, setting

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Right non-degenerate solution of Yang-Baxter equation

We have seen that with the skew brace we obtain solutions that are non-degenerate. However there are solutions $r : B \times B \to B \times B, (a, b) \mapsto (\lambda_a(b), \rho_b(a))$ the are only right non-degenerate, i.e., that have only $\rho_b \in \mathrm{Sym}_B$.

- (already seen) Let B be a group and let $f : B \to B$ be a group homomorphism such that $f^2 = f$ then the map $r : B \times B \to B \times B$ given by

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Semi-brace

F. Catino, I.C. and P. Stefanelli are working on a generalization of the skew brace that has right non degenerate solution as solution of Yang-Baxter equation associated to it.

Definition

A **(right) semi-brace** is a set B with two operations $+$ and $\circ$ such that $(B,+)$ is a right cancellative semigroup, $(B,\circ)$ is a group and

$$(a+b)\circ c=(a+c^{-})\circ c+b\circ c \tag{31}$$

holds for all $a,b,c\in B$, where c^{-} is the inverse of c respect to $\circ$.

As for radical braces and for skew braces, with this structure it is possible to construct solutions of Yang-Baxter equation but these are not necessarily left non-degenerate.

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Example

1. Let $(B, +, \circ)$ a skew brace, then B is also a semi-brace. In fact, let $a, b, c \in B$, then

$$\begin{aligned}(a + c^{-}) \circ c + b \circ c &= a \circ c - c + c \circ c^{-} + b \circ c \\ &= a \circ c - c + b \circ c \\ &= (a + b) \circ c.\end{aligned}$$

2. Let $(B, \circ)$ a group and set $a + b := a$ for all $a, b \in B$. Then $(B, +, \circ)$ is a semi-brace. In fact, clearly $(B, +)$ is a right cancellative semigroup, moreover

$$(a + c^{-}) \circ c + b \circ c = a \circ c = (a + b) \circ c,$$

for all $a, b, c \in B$. Note that if $|B| \geq 2$ then $(B, +)$ is not a group and so $(B, +, \circ)$ is not a skew brace.

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Thank you for your attention!