Bijective solutions to the pentagon equation

Ilaria Colazzo

I.Colazzo@leeds.ac.uk

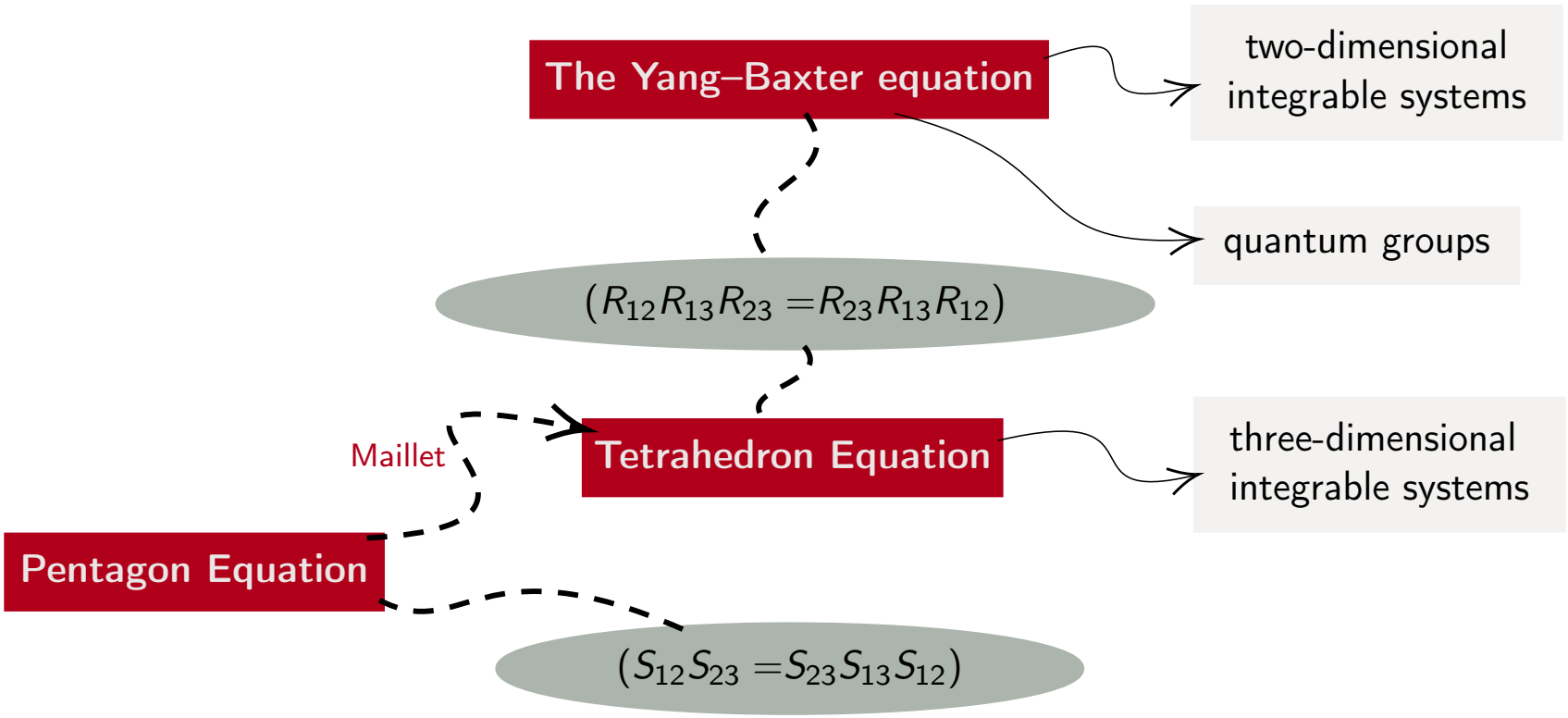
Yang–Baxter Algebras

Edinburgh (UK), 3–7/06/2024

UNIVERSITY OF LEEDS

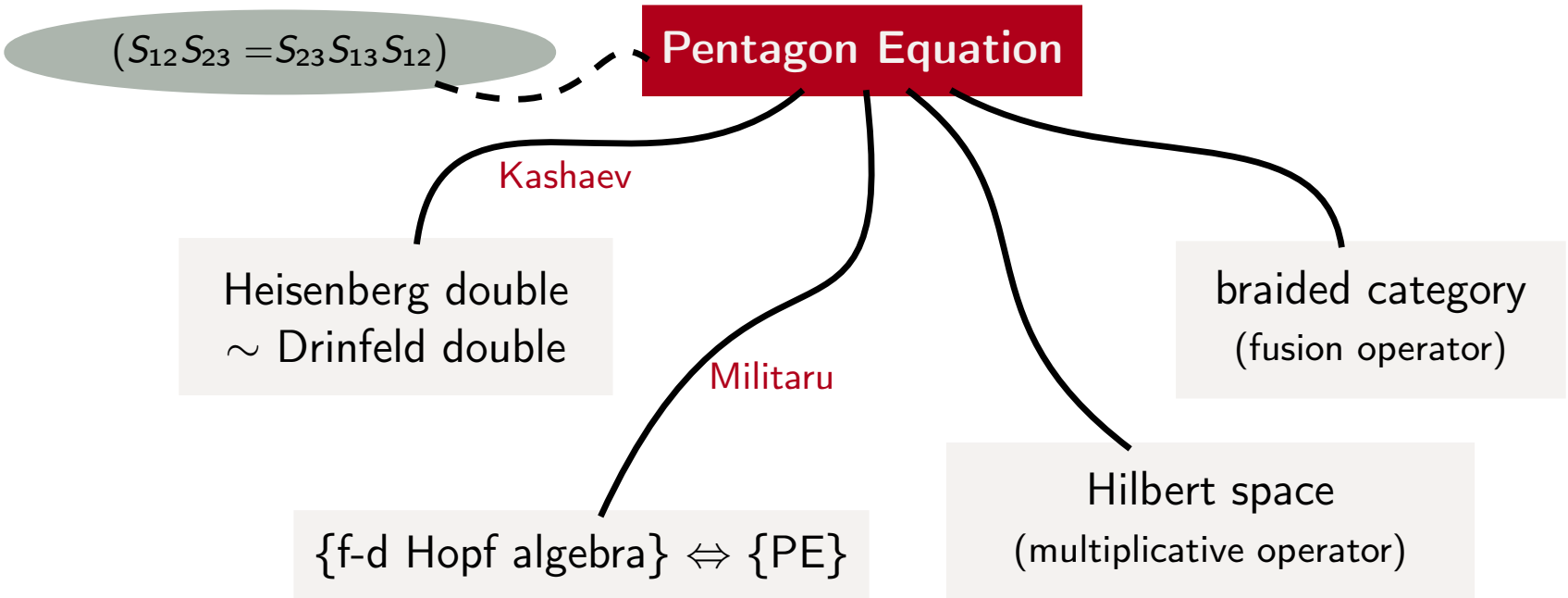
Motivation (I)

UNIVERSITY OF LEEDS



Motivation (II)

UNIVERSITY OF LEEDS



The problem and our aim

Problem

Description of set-theoretic solutions of the Pentagon Equation

(Kashaev and Reshetikhin, Jiang and Liu, Catino, Mazzotta, Miccoli and Stefanelli...)

Aim

Description of all bijective set-theoretic solutions of the Pentagon Equation

Set-theroretic solutions to the PE

UNIVERSITY OF LEEDS

The pair (S, s) is a set-theoretic solution to the Pentagon Equation if S is a set, $s : S \times S \to S \times S$ is a map such that the following is satisfied in $S \times S \times S$

$$s_{23}s_{13}s_{12} = s_{12}s_{23}$$

$$s_{23} = \mathrm{id} \times s \qquad s_{12} = s \times \mathrm{id}$$

$$s_{13} = (\tau \times \mathrm{id})(\mathrm{id} \times s)(\tau \times \mathrm{id}).$$

Set-theoretic solutions to the PE

UNIVERSITY OF LEEDS

The map (S, s) is

- finite if S is finite,
- bijective if s is bijective,
- involutive if $s^2 = \mathrm{id}$.

Write $s(x, y) = (x \cdot y, \theta_x(y))$ then

(S, s) is a solution of the PE $\iff$

1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
2. $\theta_x(y) \cdot \theta_{x \cdot y}(z) = \theta_x(y \cdot z)$
3. $\theta_{\theta_x(y)} \theta_{x \cdot y} = \theta_y$

Example

Let S be a semigroup, $f \in \mathrm{End}(S)$, $f^2 = f$, and

$$s(x, y) = (xy, f(y)).$$

Then (S, s) is a solution to the PE.

Finite, bijective solutions to the PE

UNIVERSITY OF LEEDS

Form now on we will focus on finite, bijective solutions to the PE.

Example

Let G be a group and

$$s_G(x, y) = (xy, y).$$

Then (S, s_G) is the unique bijective solution to the PE.

General facts

UNIVERSITY OF LEEDS

Fact (with Jespers and Kubat). Let (S, s) be a finite bijective solution to the PE. Then S is a completely simple semigroup.

Fact (with Van Antwerpen and Okniński). Let (S, s) be a finite bijective solution to the PE. Then

S is a left group.

$S \cong E \times G$

E is a left zero group, i.e. $xy = x,\ \forall x, y \in X$.

G is a group.

θ_x is bijective $\forall x \in S$.

General facts

Fact (with Van Antwerpen and Okniński). Let (S, s) be a finite bijective solution to the PE. Then

- Then, for every $x \in S$ there exists an $e \in E(S)$ such that $\theta_x = \theta_e$. Moreover, the set $\theta_E = \{\theta_e : e \in E(S)\}$ is a group with identity $\theta_{\theta_e}(e)$ independent of the choice of $e \in E(S)$.
- Now fix $e_0 \in E(S)$ and denote by $1 = \theta_{e_0}(e_0)$ and $G_1 = 1G$. Then for any $g \in G_1$, the map $\theta_g = \mathrm{id}_S$.

Description of involutive solutions

with Jespers and Kubat

UNIVERSITY OF LEEDS

Let (S, s) an involutive solution of the PE. Then (S, s) is isomorphic to a solution defined on $X \times A \times G$, where

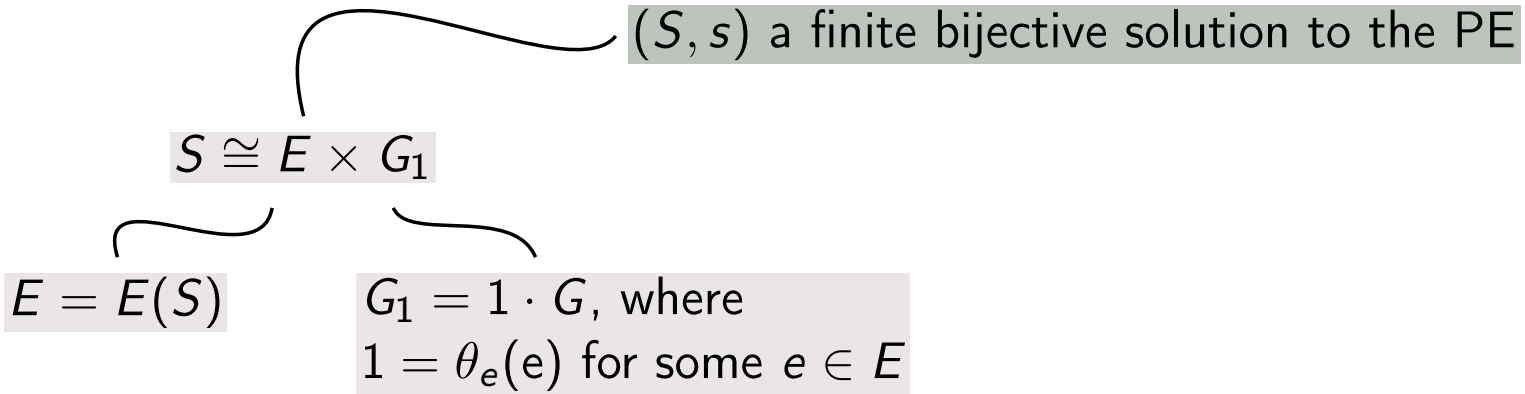
- A and G are two elementary abelian 2-groups,
- X is a set,

and

$$(\alpha, a, x)(\beta, b, y) = (\alpha, a, xy),$$
$$\theta_{(\alpha,a,x)}(\beta, b, y) = (\beta, ab, y).$$

The retraction

UNIVERSITY OF LEEDS



For $e, f \in E$ and $g, h \in G_1$ define

$$(e, g) \approx (f, h) \iff \begin{array}{l} \theta_e = \theta_f, \\ g = h. \end{array}$$

Denote $\overline{S} = S/\approx$ .Then s induces a bijective solution $(\overline{S}, \overline{s})$ called the retract of (S, s).

The retract is irretractable

Let (S, s) be a finite bijective solution and $(\overline{S}, \overline{s})$ its retract. We say (S, s) is irretractable if $S = \overline{S}$.

Fact. Let (S, s) be a finite bijective solution and $(\overline{S}, \overline{s})$ its retract. Then,

- there exists a unique idempotent element, say 1, such that $\theta_1 = \mathrm{id}$,
- $(\overline{S}, \overline{s})$ is irretractable.

How to lift a irretractable solution?

UNIVERSITY OF LEEDS

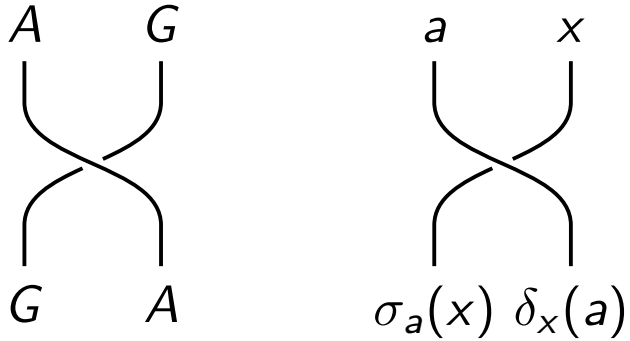
Fact (with Van Antwerpen and Okniński). Let (S, s) be a finite bijective solution and $(\overline{S}, \overline{s})$ its retract. Then there exists a set X such that (S, s) is isomorphic to the solution defined on the cartesian product $X \times \overline{S}$ such that

$$(\alpha, h)(\beta, g) = (\alpha, hg)$$
$$\theta_{(\alpha,h)}(\beta, g) = (\beta, \overline{\theta}_h(g)).$$

Therefore, to describe all finite bijective solutions is enough to describe all irretractable solutions.

From matched pairs to solutions

UNIVERSITY OF LEEDS



(G, A, σ, δ) a matched pair of groups, then $G \times A$ is a group w.r.t. $(x, a)(y, b) = (x\sigma_a(y), \delta_y(a)b)$.

Theorem (with Van Antwerpen and Okniński)

(G, A, σ, δ) a matched pair of groups S the Cartesian product $A \times G$ of A and G. Then $s : S \times S \to S \times S$ defined by

$$s((a, x), (b, y)) = ((a, xy), (b(\delta_x(a))^{-1}, \sigma_{\delta_x(a)}(y)))$$

gives an irretractable bijective solution.

From solutions to matched pairs

UNIVERSITY OF LEEDS

(S, s) bijective solution

$s_{23}s_{12} = s_{12}s_{13}s_{23}$

s^{-1} satisfies

$s_{12}^{-1}s_{23}^{-1} = s_{23}^{-1}s_{13}^{-1}s_{12}^{-1}$

write $s(x, y) = (\psi_y(x), y \circ x)$

$x \circ (y \circ z) = (x \circ y) \circ z$

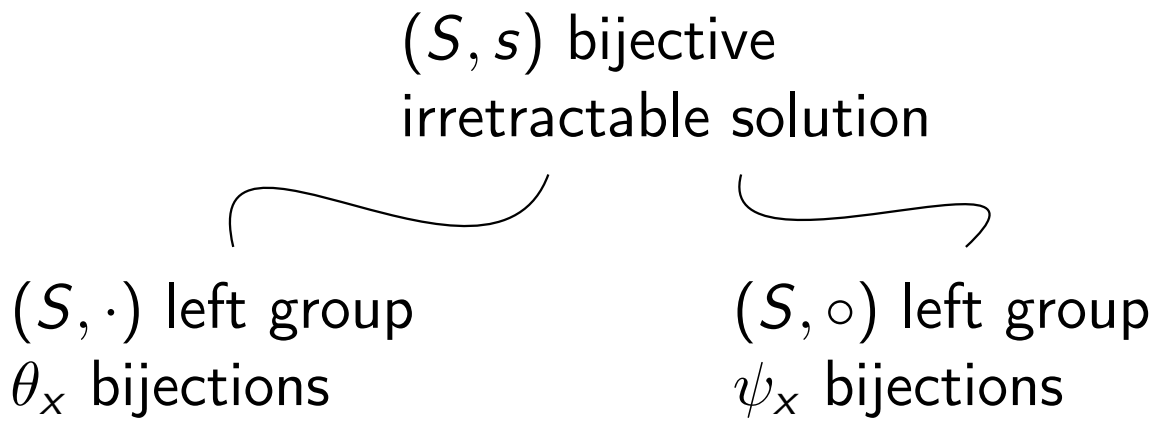
$\psi_z(y) \circ \psi_{z \circ y}(x) = \psi_z(y \circ x)$

$\psi_{\psi_z(y)}\psi_{z \circ y} = \psi_y$

$t(x, y) = (x \circ y, \psi_x(y))$ satisies the PE

From solutions to matched pairs

UNIVERSITY OF LEEDS



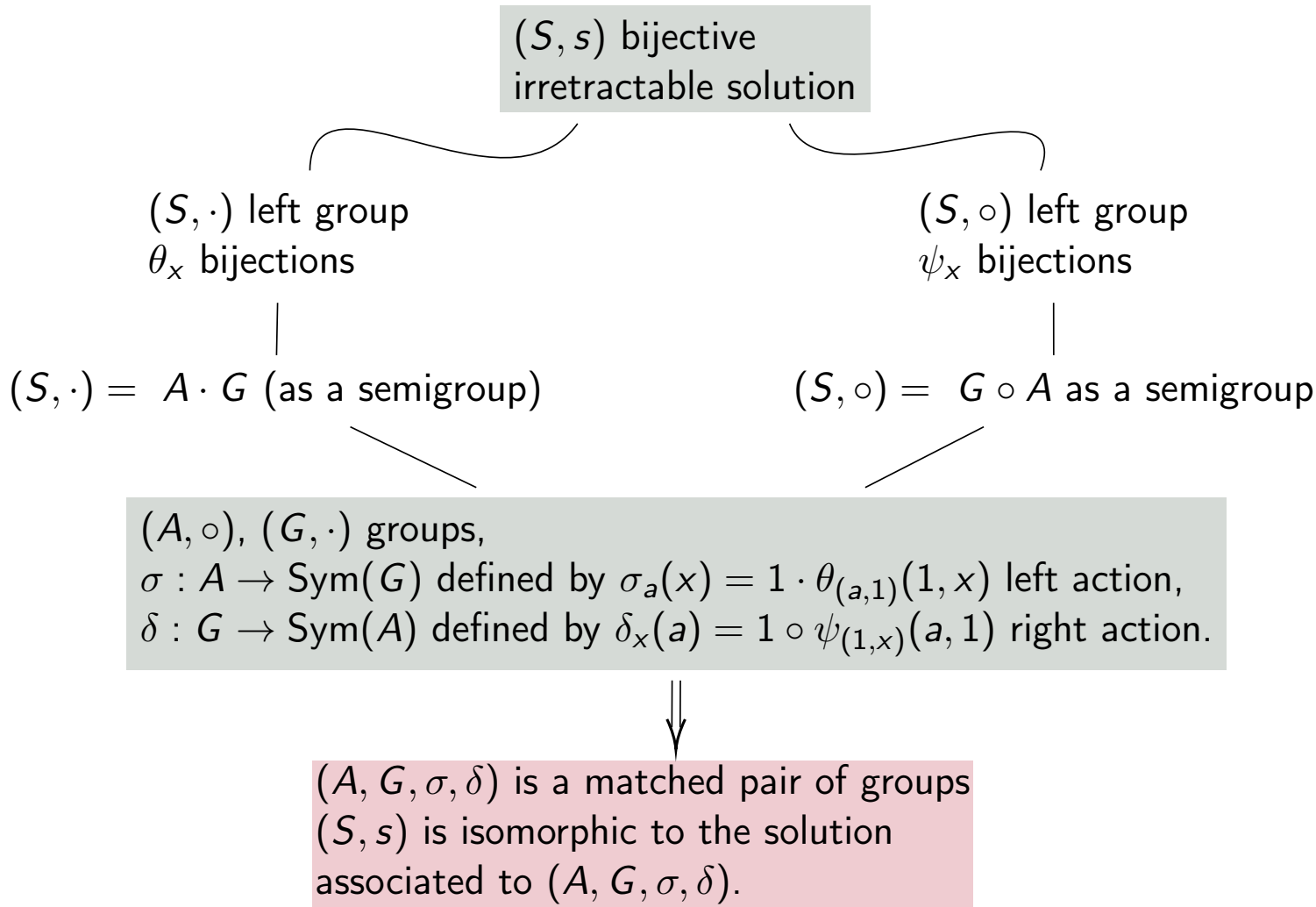
Fact. (S, s) a finite irretractable bijective solution. Then,

- $A = 1 \circ S$ is a subgroup of $(S, \circ)$ and coincides with $E(S, \cdot)$,
- $G = 1 \cdot S$ is a subgroup of $(S, \cdot)$ and coincides with $E(S, \circ)$.

From solutions to matched pairs

with Van Antwerpen and Okniński

UNIVERSITY OF LEEDS



Conclusion

UNIVERSITY OF LEEDS

Theorem (with Van Antwerpen and Okniński)

Let (S, s) be a finite bijective set-theoretic solution. Then there exist

- a matched pair of groups (A, G, σ, δ),
- a set X,

such that (S, s) is isomorphic to the solution on the Cartesian product $X \times A \times G$ such that

$$(\alpha, a, x)(\beta, b, y) = (\alpha, a, xy),$$
$$\theta_{(\alpha,a,x)}(\beta, b, y) = (\beta, b(\delta_x(a))^{-1}, \sigma_{\delta_x(a)}(y))$$

for $\alpha, \beta \in X, a, b \in A$ and $x, y \in G$.

An application

Let $(B, +, \circ)$ be a skew brace. Then

- $\lambda : (B, \circ) \to \mathrm{Aut}(B, +)$ by $\lambda_a(b) = -a + a \circ b$ is a left action,
- $\rho : (B, \circ) \to \mathrm{Sym}(B)$ by $\rho_b(a) = (\lambda_a(b))' \circ a \circ b$ is a right action.

Hence, (B, B, λ, ρ) is a matched pair of groups.
Therefore, B defines a irretractable bijective solution $(B \times B, s)$ where

$$\begin{aligned} s((a,x),(b,y)) &= ((a, x \circ y), (b \circ (\rho_x(a))', \lambda_{\rho_x(a)}(y))) \\ &= ((a, xy), (b \circ (x' \circ a - x'), (a' + x)' \circ (a' + y))). \end{aligned}$$

Thanks for your attention!