Left Semi-Braces and the Yang-Baxter equation

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Abstract

This thesis focuses on the new algebraic structure: *semi-brace*. We study basic properties of this structure and we show that semi-braces are a generalization of braces. Moreover we introduce new constructions of semi-braces, the asymmetric product and the matched product, in order to obtain several examples of semi-braces. Finally, we prove that we may construct left non-degenerate solutions of the Yang-Baxter equation through left semi-braces.

Abstract (Italian)

Questa tesi è incentrata su una nuova struttura algebrica: i *semi-braces*. Si studieranno le proprietà basilari di questa nuova struttura e si mostrerà che i semi-braces sono una generalizzazione dei braces. Inoltre si introdurranno delle costruzioni di semi-braces, il prodotto asimmetrico e il prodotto matched per ottenere altri esempi di semi-braces. Infine proveremo che è possibile ottenere tramite i semi-braces soluzioni dell'equazione di Yang-Baxter che siano solo non degeneri a sinistra.

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Introduction

The *quantum Yang-Baxter equation* (briefly QYBE) arose in some problems of statistical mechanics. First it appears in work of Yang [33], in 1967, and of Baxter [5], in 1973.

Let V be a vector space over a field, a *solution of the Yang-Baxter equation* is a linear map $\mathcal{R} : V \otimes V \to V \otimes V$ such that

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12},$$

where $\mathcal{R}_{12} := \mathcal{R} \otimes \mathrm{id}_V$, $\mathcal{R}_{23} := \mathrm{id}_V \otimes \mathcal{R}$ and $\mathcal{R}_{13} = (\mathrm{id}_V \otimes \tau)(\mathcal{R} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \tau)$ and $\tau : V \otimes V \to V \otimes V$ is the twist map, i.e., the linear map such that $\tau(u \otimes v) = v \otimes u$, for all $u, v \in V$. Moreover, $\mathcal{R} : V \otimes V \to V \otimes V$ is a solution of the QYBE if and only if the map $R := \tau\mathcal{R}$ satisfies the so-called *braid equation*

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{I}$$

Also in this case one can say that R is a solution of the Yang-Baxter equation.

In 1992, Drinfeld [15] proposed to study a simplified case, i.e., the *set-theoretical solution of the quantum Yang-Baxter equation*. In particular, fixing a basis X on the vector space V we may focus on the problem to find all solutions R, i.e., a solution of (I), induced by a linear extension of a function $r : X \times X \to X \times X$ that satisfies the *braid equation*

$$r_1 r_2 r_1 = r_2 r_1 r_2, \tag{II}$$

where $r_1 := r \times \mathrm{id}_X$ and $r_2 := \mathrm{id}_X \times r$. These maps r are called set-theoretical solutions of the QYBE, hereinafter we call these functions *solutions*.

Let X be a non-empty set. The following maps satisfy the relation (II).

1. The *twist map*, $\tau : X \times X \to X \times X$, i.e., the function defined by $\tau(x, y) = (y, x)$, for all $x, y \in X$.

2. The map $r : X \times X \to X \times X$ defined by $r(x, y) = (f(y), g(x))$, where f, g are functions from X into itself such that $fg = gf$ (see V.V. Lyubashenko's solutions in [15], Example 1 section 9).

3. The map $r : X \times X \to X \times X$ defined by $r(x, y) = (x * y, x)$, where $*$ is an operation on X such that $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in X$ (see B.B. Venkov's solutions in [15], Example 2 section 9). For instance, if X is a group and we set $x * y := xyx^{-1}$, for all $x, y \in X$ then $*$ satisfy previous condition.

4. If X is a group and $f : X \times X \to X \times X$ is a group endomorphism such that $f^2 = f$, then the map $r : X \times X \to X \times X$ defined by $r(x, y) = \left(xyf(x)^{-1}, f(x)\right)$, for all $x, y \in X$ (see [18]). Note that if $f = \mathrm{id}_X$ we obtain the previous solution in the particular case in which $*$ is the operation of conjugation.

Determining all set-theoretical solutions of the quantum Yang-Baxter equation is a very difficult task. Seminal papers of Etingof, Schedler and Soloviev [16] and of Gateva-Ivanova and M. Van den Bergh [17] laid the groundwork for the study of a particular class of solutions of the quantum Yang-Baxter equation, the *non-degenerate involutive ones*, i.e., a map $r : X \times X \to X \times X$, where X is a set, satisfies (II) such that the first and the second projections of r are bijections and $r^2 = \mathrm{id}_{X \times X}$.

Clearly, the twist map is a non-degenerate involutive set-theoretical solution of the QYBE. For instance, Lyubashenko's solution is non-degenerate and involutive if and only if f, g are bijections and $g = f^{-1}$. In the case of a solution in 4., r is non-degenerate if and only if $f = \mathrm{id}_X$ and r is involutive if and only if X is abelian and $f = \mathrm{id}_X$.

In 2007, Rump in [28] introduced the *braces* to construct solutions of QYBE. Let $(B, +)$ be an abelian group and $\cdot$ an operation on B such that

1. $a \cdot (b + c) = a \cdot b + a \cdot c$, for all $a, b, c \in B$;

2. $(a + b + a \cdot b) \cdot c = a \cdot c + b \cdot c + a \cdot (b \cdot c)$, for all $a, b, c \in B$;

3. the map $\gamma_a : B \to B, x \mapsto a \cdot x + x$ is bijective, for every $a \in B$.

Then $(B, +, \cdot)$ is called a *left brace*. If B is a left brace, then the map $r : B \times B \to B \times B$ defined by

$$r(a, b) = \left(\gamma_a(b), \gamma^{-1}_{\gamma_b(a)}(a)\right)$$

for all $a, b \in B$ is a non-degenerate involutive solution of QYBE.

Cedó, Jespers and Okniński in [13], reformulate the definition of left brace. A set B with two operations $+$, $\circ$ such that $(B,+)$ is an abelian group, $(B,\circ)$ is a group and

$$a\circ(b+c)+a=a\circ b+a\circ c, \tag{III}$$

holds for all $a,b,c\in B$, is a left brace. The link between $\circ$ and $\cdot$, $a\circ b=a\cdot b+a+b$, allows us to go from one definition to the other and conversely.

Braces and their relation with non-degenerate involutive solutions of QYBE have been widely studied, for instance, see [26], [3], [1], [9], [10].

Recently, Guarnieri and Vendramin in [19] introduce *skew left braces*, a generalization of left brace as a tool to find non-degenerate and bijective solutions, not necessarily involutive, i.e., bijective maps that satisfy (II) and such that first and second projections are bijective. Thus, skew left braces answer the problem of finding non-involutive solutions, initially studied by Lu, Yan and Zhu [25] and by Soloviev [32]. A skew left brace is a set B with two operations $+$, $\circ$ such that $(B,+)$ and $(B,\circ)$ are groups and the condition (III) is replaced by

$$a\circ(b+c)=a\circ b-a+a\circ c, \tag{IV}$$

for all $a,b,c\in B$.
If B is a skew left brace and $\lambda_a:B\to B, b\mapsto -a+a\circ b$, for every $a\in B$, then the map $r:B\times B\to B\times B$ defined by

$$r(a,b)=\left(\lambda_a(b),\lambda^{-1}_{\lambda_b(a)}\left(-(b\circ a)+b+(b\circ a)\right)\right)$$

is a bijective non-degenerate solution of QYBE, not necessarily involutive.

Catino, Stefanelli and myself in [11] introduce *left semi-braces*, a generalization of skew left braces, in order to determine new solutions r of QYBE that are only left non-degenerate, i.e, such that the first projection of r is bijective. We say that a set B with two operations $+$ and $\circ$ is a left semi-brace if $(B,+)$ is a left cancellative semigroup, $(B,\circ)$ is a group and condition (IV) is replaced by

$$a\circ(b+c)=a\circ b+a\circ\left(a^{-}+c\right), \tag{V}$$

for all $a,b,c\in B$, where we denote by a^{-} the inverse of a respect to $\circ$.
If B is a left semi-brace, then the map $r:B\times B\to B\times B$ defined by

$$r(a,b)=\left(a\circ\left(a^{-}+b\right),\left(a^{-}+b\right)^{-}\circ b\right)$$

is a solution of QYBE. Note that if B is a left semi-brace that is not a skew left brace then the associated solution is right degenerate (as shown in Remark 1.2.12).

In this thesis, we prove that not all left cancellative semigroups may be the additive structure of a left semi-brace. In particular, we prove that the additive semi-group is a right group, see Proposition 1.2.4. Further we introduce the concept of ideal, Definition 1.3.1, for a left semi-brace and we prove that if the left semi-brace is a skew left brace then the definition of ideal coincides with the definition introduced by Guarnieri and Vendramin in [19]. Moreover, we introduce new constructions, the asymmetric product (Chapter 2) and the matched product (Chapter 3), that allow us to construct more examples of left semi-brace. Finally, as corollary of matched product we may prove that any left semi-brace is the matched product of a skew left brace and a trivial semi-brace, see Proposition 3.4.1. A trivial left semi-brace is the easier one that one may construct, in fact it is a group E with additive structure the right zero-semigroup on E, i.e., with the sum given by $a + b = b$, for all $a, b \in E$.

Another aspect strictly linked with braces is their connection with regular subgroups of the holomorph of a group. Further a particular kind of brace, the *left brace over a field*, has a link with the regular subgroups of the affine group. In fact there is a one-to-one correspondence between F-braces with additive structure a fixed vector space V and regular subgroups of the affine group of V. This correspondence answers to the problem of finding all regular subgroups of the affine group posed in a seminal paper by Liebeck, Praeger and Saxl [24].

If V is a vector space over a field F, denoted by $\mathrm{T}(V)$, the group of all translations on V and by $\mathrm{GL}(V)$ the linear group, then the affine group of V is the group generated by $\mathrm{GL}(V)$ and $T(V)$. Clearly, $\mathrm{T}(V)$ and any its conjugate subgroup by an element of $\mathrm{GL}(V)$ are abelian regular subgroups of $\mathrm{AGL}(V)$. Caranti, Dalla Volta and Sala in [6] obtained a simple description of all abelian regular subgroups of the affine group $\mathrm{AGL}(V)$ in terms of commutative associative radical algebras with the underlying vector space V. Hegedűs in [22] find examples of non-abelian regular subgroup of some particular affine groups.

A very interesting result was obtained by Catino and Rizzo in [7], that established a one-to-one correspondence between regular subgroups of the affine group $\mathrm{AGL}(V)$ and *left F-braces* with underlying vector space V. We may reformulate the original definition of left F-brace in the following way: a vector space V over a field F with an operation $\circ$ is called a *left F-brace*, if $(V, +, \circ)$ is a left brace and

$$\mu(a \circ b) = a \circ (\mu b) + (\mu - 1)\, a, \tag{VI}$$

for all $\mu \in F$ and $a, b \in V$. Catino and Rizzo proved that if $(V, +, \circ)$ is a left brace over a field F, then the set

$$\{(a, \lambda_a) \mid a \in V\},$$

where $\lambda_a : V \to V, x \mapsto -a + a \circ b$, for every $a \in V$, is a regular subgroup of the affine group $\mathrm{AGL}(V)$. Conversely, if $T = \{(a, \phi_a) \mid a \in V\}$ is a regular subgroup of $\mathrm{AGL}(V)$ and set $a \circ b := a + \phi_a(b)$, for all $a, b \in V$, then the vector space V with the operation $\circ$ is a left F-brace. By this result, the problem of determining all regular subgroups of an affine group may be replaced by that of determining all left F-braces. In this thesis we obtain as corollary of the asymmetric product of left semi-brace, the asymmetric product of left F-brace Corollary 2.4.1 and Corollary 2.4.4, already introduced by Catino, Stefanelli and myself in [10] that allows us to put in a more general context the example of regular subgroup with trivial intersection with the translation group introduced by Hegedűs (see Theorem 4.3.15). Moreover we provide a complete description of all left F-braces with non-trivial annihilator, the Hochschild product of left F-braces (Proposition 1.3.36), already introduced by Catino, Stefanelli and myself in [9], that allows us to describe regular subgroup of the affine group with non-trivial intersection with the translation group. In this thesis we obtain the Hochschild Product of left F-braces as a corollary of a new construction of skew left brace, the *extension of skew left brace* (see Theorem 1.3.21) a generalization of the analogue construction introduced by Bachiller in [2] for left braces.

Guarnieri and Vendramin in [19], generalize the correspondence between regular subgroups of the affine group $\mathrm{AGL}(V)$ and left F-brace with underlying vector space V to a correspondence between regular subgroups of the holomorph of a group $\mathrm{Hol}(B)$ and left skew brace with additive group B. If B is a group, then the *holomorph* $\mathrm{Hol}(B)$ is the semidirect product between the group B and the group $\mathrm{Aut}(B)$ of automorphisms of B where the product is given by

$$(a, \alpha)(b, \beta) = (a + \alpha(b), \alpha\beta),$$

for all $a, b \in B$ and $\alpha, \beta \in \mathrm{Aut}(B)$.

In this thesis we generalize the definition of the holomorph of a group to a definition of a holomorph of a right group, Definition 4.3.1. In particular, if B is a right group we say that the *holomorph of* B is the semigroup $\mathrm{Hol}(B)$ defined over the cartesian product of B and $\mathrm{Aut}(B)$, the group of automorphisms of B, where the product is given, as in the classical case, by

$$(a, \alpha)(b, \beta) = (a + \alpha(b), \alpha\beta),$$

for all $a, b \in B$ and $\alpha, \beta \in \mathrm{Aut}(B)$. Further we prove a correspondence (Theorem 4.3.5) between the regular subgroups of the holomorph of a right group B and a left semi-brace with additive semigroup B.

Thesis Structure

In **Chapter 1**, we introduce the definition and first basic examples of *left semi-braces*, see Section 1.1. We study the additive structure of a left semi-brace, in particular we prove in Proposition 1.2.4 that the additive semigroup of a left semi-brace is a right group, i.e., it is the direct sum of a group and a right zero-semigroup. Moreover we introduce the concept of ideal, Definition 1.3.1, for left semi-braces that allows us to obtain new left semi-braces as quotient of known ones. Further we focus on particular type of ideals, the socle and the annihilator. We show that if we have a skew left brace, then the definition of ideal is the same introduced in [19], Proposition 1.3.4. Finally, we extend the correspondence between left braces and bijective 1-cocycles of an abelian group to a correspondence between left semi-braces and bijective 1-cocycles of a right group, Theorem 1.4.4.

In **Chapter 2** and **Chapter 3** we introduce constructions of left semi-braces: the asymmetric product, Theorem 2.1.4, and the matched product, Theorem 3.1.1. These constructions allow us to obtain new left semi-braces. In particular, by matched product we prove that every left semi-braces can be seen as a product of a trivial left semi-brace and a skew left brace, Corollary 3.4.2.

In **Chapter 4** we prove that from every left semi-brace we may obtain a set-theoretical solution of the QYBE, Theorem 4.2.1. In particular, we have that these solutions are left non-degenerate but, in general, not right non-degenerate. Finally, we extend the correspondence between skew left braces and regular subgroups of the holomorph of a group to a correspondence between left semi-braces and regular subgroups of the holomorph of a right group, Theorem 4.3.5. In particular, we give a definition of the holomorph of a right group, see Definition 4.3.1, and we prove that every regular subgroup of the holomorph of a right group, $(B, +)$, is the multiplicative group of a left semi-brace that has additive structure exactly B. On the other hand, if we have a left semi-brace then its multiplicative group is isomorphic to a regular subgroup of the holomorph of its additive semigroup.

1 Left semi-braces

1.1 Definitions and Examples

We introduce the definition and basic examples of left semi-brace given in [11].

Definition 1.1.1. Let B be a set with two operations $+$ and $\circ$ such that $(B,+)$ is a left cancellative semigroup and $(B,\circ)$ is a group. We say that $(B,+,\circ)$ is a *left semi-brace* if

$$a\circ(b+c)=a\circ b+a\circ(a^{-}+c) \tag{1.1}$$

holds for all $a,b,c\in B$, where a^{-} is the inverse of a respect to $\circ$.

Remark 1.1.2. In a left semi-brace $(B,+,\circ)$, the identity 0 of the group $(B,\circ)$ is an idempotent of the semigroup $(B,+)$. In fact,

$$0+0=0\circ(0+0)=0\circ 0+0\circ(0^{-}+0)=0+0\circ(0+0)=0+0+0,$$

and by left cancellativity $0=0+0$.
In general, if $(B,+)$ is a left cancellative semigroup, then an element e of B is idempotent if and only if e is a left identity. In fact, if $e\in B$ is an idempotent and $b\in B$, then $e+e+b=e+b$ and by the left cancellativity $b=e+b$. Conversely if $e\in B$ is a left identity then in particular $e+e=e$ and so e is an idempotent. Therefore 0 is a left identity of the left semigroup $(B,+)$.

We may provide some first examples of left semi-braces.

Example 1.1.3. If $(E,\circ)$ is a group, then $(E,+,\circ)$, where $a+b=b$ for all $a,b\in E$, is a left semi-brace. In fact, first $(E,+)$ is clearly a left cancellative semigroup and if $a,b,c\in E$, then

$$a\circ b+a\circ(a^{-}+c)=a\circ c=a\circ(b+c)\,,$$

i.e., the condition (1.1) holds. We call this left semi-brace the *trivial left semi-brace* on the group $(E,\circ)$.

Example 1.1.4. Let $(B, \circ)$ be a group and f be an endomorphism of $(B, \circ)$ such that $f^2 = f$. Set

$$a + b := b \circ f(a)$$

for all $a, b \in B$, then $(B, +, \circ)$ is a left semi-brace. In fact, first, if $a, b, c \in B$ such that $a + b = a + c$, then $b \circ f(a) = c \circ f(a)$, i.e., $b = c$. Moreover,

$$\begin{aligned} a \circ b + a \circ (a^- + c) &= a \circ b + a \circ (c \circ f(a^-)) = a \circ (c \circ f(a^-)) \circ f(a \circ b) \\ &= a \circ (c \circ f(b)) = a \circ (b + c), \end{aligned}$$

for all $a, b, c \in B$. Therefore, $(B, +, \circ)$ is a left semi-brace.
On the other hand, if $(B, +, \circ)$ is a left semi-brace and $f : B \to B$ is such that $a + b = b \circ f(a)$, then f if an endomorphism of the group $(B, \circ)$ such that $f^2 = f$. In fact, if $a \in B$, then

$$\begin{aligned} f(a^-) \circ f(a) &= a + f(a^-) = a + a \circ (a^- \circ f(a^-)) = a \circ 0 + a \circ (a^- + a^-) \\ &= a \circ (0 + a^-) = a \circ a^- = 0 \end{aligned}$$

thus $f(a^-) = f(a)^-$. Further, if $a, b \in B$, then

$$\begin{aligned} a \circ f(b) &= a \circ 0 \circ f(b) = a \circ (b + 0) = a \circ b + a \circ (a^- + 0) \\ &= a \circ b + a \circ 0 \circ f(a^-) = a \circ f(a^-) \circ f(a \circ b) \\ &= a \circ f(a)^- \circ f(a \circ b) \end{aligned}$$

and so, since $(B, \circ)$ is a group, it follows that $f(a) \circ f(b) = f(a \circ b)$.
Finally,

$$f^2(a) = f(f(a)) = f(0 \circ f(a)) = f(a + 0) = a + 0 + 0 = a + 0 = f(a).$$

Left semi-braces generalize skew left braces introduced in [19].

Definition 1.1.5 (Definition 1.1 in [19])**.** Let B be a set with two operations $+$ and $\circ$ such that $(B, +)$ and $(B, \circ)$ are groups. We say that $(B, +, \circ)$ is a *skew left brace* if

$$a \circ (b + c) = a \circ b - a + a \circ c. \tag{1.2}$$

holds for all $a, b, c \in B$, where $-a$ is the inverse of a respect to $+$.

If B is a skew left brace, then $(B, +)$ is a group and, in particular, a left cancellative semigroup. Moreover if $a, b, c \in B$, then

$$a \circ b + a \circ (a^- + c) = a \circ b + a \circ a^- - a + a \circ c = a \circ b - a + a \circ c = a \circ (b + c),$$

i.e., condition (1.1) in Definition 1.1.1 holds. Hence B is also a left semi-brace.

Example 1.1.6. Let $(B, \circ)$ be a group. Set $a+b := a \circ b$ for all $a, b \in B$, then $(B, +, \circ)$ is a skew left brace. In fact, if $a, b, c \in B$, then

$$a \circ b - a + a \circ c = a \circ b \circ a^{-} \circ a \circ c = a \circ b \circ c = a \circ (b + c).$$

Hereinafter, we call this skew left brace the *zero left semi-brace* or *zero skew left brace*.

Example 1.1.7 (Example 1.3 in [19])**.** Let $(B, \circ)$ be a group. Set $a + b := b \circ a$, for all $a, b \in B$, then $(B, +, \circ)$ is a skew left brace. In fact, if $a, b, c \in B$, then

$$a \circ b - a + a \circ c = a \circ b + a^{-} + a \circ c = a \circ c \circ a^{-} \circ a \circ b = a \circ c \circ b = a \circ (b + c).$$

Skew left braces generalize left braces introduced by Rump in [28] and later formulate in an equivalent definition by Cedó, Jespers and Okniński in [13] as in the following definition.

Definition 1.1.8. Let B be a set with two operations $+$ and $\circ$ such that $(B, +)$ is an abelian group and $(B, \circ)$ is a group. We say that $(B, +, \circ)$ is a *left brace* if

$$a \circ (b + c) + a = a \circ b + a \circ c. \tag{1.3}$$

holds for all $a, b, c \in B$.

Example 1.1.9. Let R be a radical ring and consider $a \circ b := ab + a + b$, for all $a, b \in R$ the adjoint operation. Then $(R, +, \circ)$ is a left brace. In fact $(R, \circ)$ is a group and if $a, b, c \in R$, then

$$\begin{aligned} a \circ (b + c) &= a(b + c) + a + b + c = ab + ac + a + b + c \\ &= ab + a + b - a + ac + a + c = a \circ b - a + a \circ c, \end{aligned}$$

i.e., the condition (1.3) in Definition 1.1.8 holds.

Example 1.1.10. Let $(\mathbb{Z}, +)$ be the additive group of integers and consider the multiplication given by

$$a \circ b := a + (-1)^{a} b$$

for all $a, b \in \mathbb{Z}$. Then $(\mathbb{Z}, +, \circ)$ is a left brace. In fact, $(\mathbb{Z}, +)$ is an abelian group, $(\mathbb{Z}, \circ)$ is a group and if $a, b, c \in \mathbb{Z}$, then

$$\begin{aligned} a \circ (b + c) + a &= a + (-1)^{a}(b + c) + a = a + (-1)^{a} b + a + (-1)^{a} c \\ &= a \circ b + a \circ c, \end{aligned}$$

i.e., condition (1.3) in Definition 1.1.8 holds.

Example 1.1.11. Let $(\mathbb{Z}/\mathbb{Z}6, +)$ be the cyclic group of 6 elements and consider the multiplication given by

$$a \circ b := \begin{cases} a+b & \text{if } a \in \{0,2,4\} \\ a+5b & \text{if } a \in \{1,3,5\} \end{cases}$$

for al $a, b \in \mathbb{Z}/\mathbb{Z}6$. Then $(\mathbb{Z}/\mathbb{Z}6, +, \circ)$ is a left brace. In fact, $(\mathbb{Z}/\mathbb{Z}6, +)$ is an abelian group, $(\mathbb{Z}/\mathbb{Z}6, \circ)$ is a group with identity 0 and

$$a \circ (b+c) + a = \begin{cases} a+b+c+a & \text{if } a \in \{0,2,4\} \\ a+5\,(b+c)+a & \text{if } a \in \{1,3,5\} \end{cases} = a \circ b + a \circ c,$$

for all $a, b, c \in \mathbb{Z}/\mathbb{Z}6$. Hence it is a left brace (see Example 3 in [28]).

Following Rump's terminology, in [31], we may introduce a brace over a field, initially introduced as *circle algebras* in [7].

Definition 1.1.12. Let F be a field, $(B, +)$ a vector space over F and $\circ$ an operation over B such that $(B, \circ)$ is a group. We say that $(B, +, \circ)$ is a *left F-brace* (or a *left brace over the field F*) if

$$a \circ (b+c) + a = a \circ b + a \circ c \tag{1.4}$$

$$\mu\,(a \circ b) = a \circ (\mu b) + (\mu - 1)\,a \tag{1.5}$$

hold for all $a, b, c \in B$ and for every $\mu \in F$.

Example 1.1.13. Let F be a field and B a radical algebra over F and consider $a \circ b := ab + a + b$, for all $a, b \in B$. Then $(B, +, \circ)$ is a left F-brace. In fact by Example 1.1.9, $(B, +, \circ)$ is a left brace, moreover if $a, b \in B$ and $\mu \in F$, then

$$\begin{aligned} \mu\,(a \circ b) &= \mu\,(ab + a + b) = \mu\,(ab) + \mu a + \mu b = a\,(\mu b) + a + \mu b + (\mu - 1)\,a \\ &= a \circ (\mu b) + (\mu - 1)\,a, \end{aligned}$$

i.e., condition (1.5) in Definition 1.1.12 holds.

Example 1.1.14. Let F be a field and $(V, +)$ a vector space over F. Set $a \circ b = a + b$ for all $a, b \in V$, then $(V, +, \circ)$ is a left F-brace. In fact, by Example 1.1.6 $(V, +, \circ)$ is a skew left brace and since $(V, +)$ is an abelian group it is a left brace, moreover, if $a, b \in V$ and $\mu \in F$, then

$$\mu\,(a \circ b) = \mu\,(a + b) = \mu a + \mu b = a + (\mu b) + (\mu - 1)\,a = a \circ (\mu b) + (\mu - 1)\,a.$$

Hereinafter, we call this left F-brace the *zero left F-brace*.

1.2 The additive structure of a left semi-brace

The following proposition allows us to focus on the additive semigroup of a left semi-brace. In particular, we describe what kind of left cancellative semigroups with idempotents may be this additive structure.

Proposition 1.2.1. *Let $(B,+,\circ)$ be a left semi-brace and $a \in B$. Then, the map*

$$\lambda_a : B \longrightarrow B,\ b \mapsto a \circ (a^- + b)$$

is an automorphism of the semigroup $(B,+)$ and $\lambda_a^{-1} = \lambda_{a^-}$. Moreover, the function λ from the group $(B,\circ)$ into the group of the automorphisms of $(B,+)$ given by $\lambda(b) = \lambda_b$, for every $b \in B$, is a homomorphism.

Proof. Let $a \in B$. Then

$$\begin{aligned}\lambda_a(x+y) &= a \circ (a^- + x + y) = a \circ (a^- + x) + a \circ (a^- + y)\\ &= \lambda_a(x) + \lambda_a(y),\end{aligned}$$

for all $x, y \in B$. Moreover,

$$\begin{aligned}\lambda_{a\circ b}(x) &= (a \circ b) \circ \left((a \circ b)^- + x\right) = a \circ \left(b \circ b^- \circ a^- + b \circ (b^- + x)\right)\\ &= a \circ (a^- + \lambda_b(x)) = \lambda_a(\lambda_b(x)).\end{aligned}$$

Finally, $\lambda_0(x) = 0 \circ (0^- + x) = x$, for every $x \in B$, and so $\lambda_a\lambda_{a^-} = \lambda_{a\circ a^-} = \mathrm{id}_B = \lambda_{a^-\circ a} = \lambda_{a^-}\lambda_a$. Therefore, λ is a homomorphism from the group $(B,\circ)$ into the group $\mathrm{Aut}(B,+)$ of the automorphisms of $(B,+)$. □

Now, we may prove that the additive semigroup of a left semi-brace is exactly a right group.

Definition 1.2.2 (see [14], p. 37)**.** A semigroup $(B,+)$ is said a *right group* if B is a left cancellative semigroup such that, for all $x, y \in B$, there exists $t \in B$ such that $x + t = y$.

Example 1.2.3. Let B be a set. Define $a + b := b$, for all $a, b \in B$. Then $(B,+)$ is a right group, called *right zero-semigroup*.

Proposition 1.2.4. *Let $(B,+,\circ)$ be a left semi-brace. Then the additive semigroup $(B,+)$ is a right group.*

Proof. By definition $(B,+)$ is a left cancellative semigroup. Moreover if $x,y\in B$, setting $t:=\lambda_x(x^-\circ y)$, we have

$$\begin{aligned}x+t&=x+\lambda_x\left(x^-\circ y\right)=x+x\circ\left(x^-+x^-\circ y\right)=x\circ\left(0+x^-\circ y\right)\\&=x\circ x^-\circ y=y,\end{aligned}$$

since 0, the identity of the group $(B,\circ)$, is a left identity of the semigroup $(B,+)$. Therefore, $(B,+)$ is right group. □

Proposition 1.2.5 (see, for instance, Theorem I.1.27 in[14])**.** *Let $(B,+)$ be a semigroup. The following statements are equivalent:*

1. *B is a right group;*
2. *There exist a group G and a right zero-semigroup E such that B is the direct sum of G and E, i.e., $B=G+E$.*

Hereinafter, when we refer to idempotents of a semi-brace B we mean idempotents of the additive semigroup, $(B,+)$.

Example 1.2.6. Let B be the left semi-brace in Example 1.1.4. Note that the set of idempotents is $\ker f$ and the group $G=B+0$ is $\operatorname{Im} f$. In fact, if $a\in\ker f$, then $a+b=b\circ f\left(a\right)=b\circ 0=b$, for every $b\in B$, i.e., $a\in E$; conversely if $e\in E$, then $b=e+b=b\circ f\left(e\right)$, for every $b\in B$, i.e., $e\in\ker f$. Moreover, if $b\in\operatorname{Im} f$, then there exists $a\in B$ such that $b=f\left(a\right)=0\circ f\left(a\right)=a+0\in B+0$; on the other hand, if $b\in B$, then $b+0=0\circ f\left(b\right)=f\left(b\right)\in\operatorname{Im} f$.

Let us introduce basic properties related to the set E of all idempotents of a left semi-brace B and the group $G:=B+0$.

Proposition 1.2.7. *Let B be a left semi-brace, E the set of idempotents of $(B,+)$, 0 the identity of $(B,\circ)$ and $G:=B+0$. The following hold:*

1. *G is a subgroup of $(B,\circ)$, $\lambda_g(b)=-g+g\circ b$ for all $b\in B$ and $g\in G$, and $(G,+,\circ)$ is a skew left brace;*
2. *E is a subgroup of $(B,\circ)$ and $(E,+,\circ)$ is a trivial left semi-brace.*

Proof. 1. Trivially, G is not empty since $0\in G$. At first, if $g\in G$ and $b\in B$, then

$$\begin{aligned}\lambda_g(b)&=g\circ\left(g^-+b\right)=0+g\circ\left(g^-+b\right)=-g+g+g\circ\left(g^-+b\right)\\&=-g+g\circ\left(0+b\right)=-g+g\circ b.\end{aligned}$$

Moreover, if $a, b \in B$, we have

$$\begin{aligned}(a+0)\circ(b+0) &= (a+0)\circ b + \lambda_{a+0}(0) = (a+0)\circ b - (a+0) + (a+0)\circ 0\\ &= (a+0)\circ b + 0 \in B + 0 = G.\end{aligned}$$

Furthermore, if $g \in G$, then

$$g\circ(g^- + 0) = g\circ g^- + \lambda_g(0) = g\circ g^- - g + g\circ 0 = 0.$$

So, $g^- = g^- + 0 \in G$. Therefore, G is a multiplicative subgroup of B. Moreover,

$$g_1\circ(g_2+g_3) = g_1\circ g_2 + \lambda_{g_1}(g_3) = g_1\circ g_2 - g_1 + g_1\circ g_3,$$

for all $g_1, g_2, g_3 \in G$. Hence, $(G, +, \circ)$ is a skew left brace.
2. Trivially, E is not empty because $0 \in E$. Now, if $e_1, e_2 \in E$, then

$$e_1^-\circ e_2 = e_1^-\circ(e_2+e_2) = e_1^-\circ e_2 + e_1^-\circ(e_1+e_2) = e_1^-\circ e_2 + e_1^-\circ e_2.$$

Hence, $e_1^-\circ e_2 \in E$ and so $(E, \circ)$ is a multiplicative subgroup of B. Clearly, $(E, +, \circ)$ is a trivial left semi-brace. □

Corollary 1.2.8. *Let $(B, +, \circ)$ be a skew left brace. Then B is a left semi-brace. On the other hand, if $(B, +, \circ)$ is a left semi-brace with $(B, +)$ a group, then B is a skew left brace.*

Proof. Clearly $(B, +)$ is a left cancellative semigroup. Moreover if $a, b, c \in B$, then by Proposition 1.2.7 and by condition (1.2) in Definition 1.1.5 we have

$$a\circ b + a\circ(a^- + c) = a\circ b + \lambda_a(c) = a\circ b - a + a\circ c = a\circ(b+c).$$

Hence B is a left semi-brace.
Conversely, if $(B, +, \circ)$ is a left semi-brace with $(B, +)$ a group. We have only to check that (1.2) in Definition 1.1.5 holds, if $a, b, c \in B$ then

$$\begin{aligned}a\circ b - a + a\circ c &= a\circ b - a + a\circ(0+c) = a\circ b - a + a + a\circ(a^- + c)\\ &= a\circ b + 0 + a\circ(a^- + c) = a\circ(b+c),\end{aligned}$$

hence $(B, +, \circ)$ is a skew left brace □

In addition to the functions defined in Proposition 1.2.1, we introduce the functions ρ_b. These ones play a basic role in the study of a left semi-brace.

Proposition 1.2.9. *Let B be a left semi-brace and let*

$$\rho_b : B \longrightarrow B,\ a \mapsto (a^- + b)^- \circ b,$$

for every $b \in B$. Then, the map ρ from the group $(B, \circ)$ into the monoid B^B of the maps from B into B given by $\rho(b) = \rho_b$ is a semigroup antihomomorphism.

Proof. First note that, if b, c, x are in B, then

$$\begin{aligned} c \circ \left(c^- \circ \left(x^- + c\right) + b\right) &= c \circ c^- \circ \left(x^- + c\right) + c \circ \left(c^- + b\right) = x^- + \left(c + c \circ \left(c^- + b\right)\right) \\ &= x^- + c \circ (0 + b) = x^- + c \circ b \end{aligned}$$

and thus

$$\left(x^- + c \circ b\right)^- \circ c = \left(c^- \circ \left(x^- + c\right) + b\right)^- = \left(\left(\rho_c\left(x\right)\right)^- + b\right)^-.$$

Therefore,

$$\rho_{c \circ b}(x) = \left(x^- + c \circ b\right)^- \circ c \circ b = \left(\left(\rho_c\left(x\right)\right)^- + b\right)^- \circ b = \rho_b(\rho_c(x)),$$

and so ρ is a semigroup antihomomorphism from the group $(B, \circ)$ into the monoid B^B. □

If B is a skew left brace, then ρ_b is bijective for every $b \in B$ and it can be expressed in terms of λ_a functions, as the following result shows.

Corollary 1.2.10. *Let B be a skew left brace. Then*

$$\rho_a\left(b\right) = \lambda^{-1}_{\lambda_b(a)}\left(-\left(b \circ a\right) + b + \left(b \circ a\right)\right),$$

for all $a, b \in B$. In particular if B is a left brace, then

$$\rho_a\left(b\right) = \lambda^{-1}_{\lambda_b(a)}\left(b\right).$$

Proof. If $a, b \in B$, then

$$\begin{aligned} &\lambda^{-1}_{\lambda_b(a)}\left(-\left(b \circ a\right) + b + \left(b \circ a\right)\right) = \lambda_{(\lambda_b(a))^-}\left(-\left(b \circ a\right) + b + \left(b \circ a\right)\right) \\ &\lambda_{(b \circ (b^- + a))^-}\left(-\left(b \circ a\right) + b + \left(b \circ a\right)\right) \\ &= \left(b \circ \left(b^- + a\right)\right)^- \circ \left(b \circ \left(b^- + a\right) - \left(b \circ a\right) + b + \left(b \circ a\right)\right) \\ &= \left(b^- + a\right)^- \circ b^- \circ \left(-b + b \circ a - \left(b \circ a\right) + b + \left(b \circ a\right)\right) \\ &= \left(b^- + a\right)^- \circ b^- \circ b \circ a = \rho_a\left(b\right). \end{aligned}$$

In particular, if B is a left brace, then $(B,+)$ is abelian and so

$$\rho_a(b) = \lambda^{-1}_{\lambda_b(a)}\left(-(b\circ a)+b+(b\circ a)\right) = \lambda^{-1}_{\lambda_b(a)}(b),$$

for all $a,b\in B$. □

We introduce some important tools that are useful in the study of left semi-braces.

Proposition 1.2.11. *Let B be a left semi-brace, E the set of idempotents of $(B,+)$, 0 the identity of $(B,\circ)$ and $G := B+0$. The following hold:*

1. *$\lambda_b(E) = E$, for every $b\in B$;*
2. *$\rho_a(B) = G$, for every $a\in B$;*
3. *$\rho_{a|_G} : G\to G$ is bijective, for every $a\in B$;*
4. *$b\in G$ if and only if $\lambda_b(0)=0$;*
5. *$b\in E$ if and only if $\rho_c(b^-)=0$, for every $c\in B$;*
6. *$b\circ G = G+b$, for every $b\in B$;*
7. *$b+E = b\circ E$, for every $b\in B$.*

Proof. 1. Let $b\in B$ and $e\in E$. Then, by Proposition 1.2.1,

$$\lambda_b(e)+\lambda_b(e) = \lambda_b(e+e) = \lambda_b(e).$$

Hence, $\lambda_b(e)\in E$. On the other hand, by the surjectivity of λ_b, there exists $t\in B$ such that $\lambda_b(t)=e$. Moreover, by Proposition 1.2.1,

$$t = \lambda_0(t) = \lambda_{b^-\circ b}(t) = \lambda_{b^-}(\lambda_b(t)) = \lambda_{b^-}(e)\in E,$$

and therefore $\lambda_b(E)=E$.
2. First of all, let us note that if $e\in E$ and $b\in B$, by 1. in Proposition 1.2.7

$$\begin{aligned}\rho_e(b) &= (b^-+e)^-\circ e = (e^-\circ(b^-+e))^- = (e^-\circ b^- + e^-\circ e)^-\\ &= (e^-\circ b^- + 0)^- \in G.\end{aligned}$$

Now, let $a \in B$. Then there exist $g \in G$ and $e \in E$ such that $a = g + e = g \circ \lambda_{g^-}(e)$. If $b \in B$, since by 1. $\lambda_{g^-}(e) \in E$, we have, by the previous relation and by Proposition 1.2.9, that

$$\rho_a(b) = \rho_{g\circ\lambda_{g^-}(e)}(b) = \rho_{\lambda_{g^-}(e)}(\rho_g(b)) \in G.$$

On the other hand, if $g \in G$, setting $b := \rho_{a^-}(g)$, we have

$$\rho_a(b) = \rho_a(\rho_{a^-}(g)) = \rho_0(g) = (g^- + 0)^- \circ 0 = (g^- + 0)^- = g.$$

3. Let $a \in B$. If $g \in G$, then $b := \rho_{a^-}(g) \in G$ and $g = \rho_a(b)$. Moreover, let $h, k \in G$ such that $\rho_a(h) = \rho_a(k)$. Then,

$$h = \rho_0(h) = \rho_{a^-}(\rho_a(h)) = \rho_{a^-}(\rho_a(k)) = \rho_0(k) = k.$$

4. Let $b \in G$. Then, by 1. in Proposition 1.2.7, $\lambda_b(0) = -b + b \circ 0 = 0$. Conversely, if $b \in B$ such that $\lambda_b(0) = 0$, then $0 = \lambda_b(0) = b \circ (b^- + 0)$, and so $b^- = b^- + 0 \in G$. Therefore, $b \in G$.

5. If $b \in E$ and $c \in B$, then $\rho_c(b^-) = (b + c)^- \circ c = c^- \circ c = 0$. Conversely, let $b \in B$ such that $\rho_c(b^-) = 0$, for every $c \in B$. In particular, we have $0 = \rho_b(b^-) = (b+b)^- \circ b$ and thus $b + b = b$.

6. Let $x \in b \circ G$ and $g \in G$ such that $x = b \circ g$. We have, by 2.,

$$\begin{aligned} x = b \circ g &= b \circ (g^-)^- = b \circ (\rho_0(g^-))^- = b \circ (\rho_b(\rho_{b^-}(g^-)))^- \\ &= b \circ \left(\left(\rho_{b^-}(g^-)^- + b\right)^- \circ b\right)^- = b \circ b^- \circ \left(\rho_{b^-}(g^-)^- + b\right) \\ &= \rho_{b^-}(g^-)^- + b \in G + b. \end{aligned}$$

On the other hand, if $x \in G + b$ and $g \in G$ are such that $x = g + b$, then, by 2.,

$$x = g + b = b \circ b^- \circ (g + b) = b \circ \left((g + b)^- \circ b\right)^- = b \circ (\rho_b(g^-))^- \in b \circ G.$$

7. If $b \in B$ and $e \in E$, by 1., we have

$$b \circ e = b \circ \lambda_{b^-}(\lambda_b(e)) = b + \lambda_b(e) \in b + E.$$

On the other hand, if $b \in B$ and $e \in E$, then

$$b + e = b \circ b^- \circ (b + e) = b \circ \lambda_{b^-}(e) \in b \circ E.$$

□

Remark 1.2.12. Let B be a left semi-brace and E the set of idempotents of B. By 2. in the previous proposition, we have that if $|E| \geq 2$, clearly ρ_a is not bijective and so ρ is a semigroup antihomomorphism that is not a monoid antihomomorphism.

1.3 Ideals, socle and annihilator of a left semi-brace

In this section we introduce the concept of ideal of a left semi-brace in order to obtain new left semi-braces. Moreover, we focus on a particular ideal, the *socle* of a left semi-brace, which is a generalization of that already introduced by Rump for braces [28] and then by Guarnieri and Vendramin for skew left braces [19].

Definition 1.3.1. Let B be a left semi-brace, E the set of idempotents of $(B,+)$, 0 the identity of $(B,\circ)$ and $G := B + 0$. We say that a subsemigroup I of $(B,+)$ is an *ideal* if the following hold:

1. I is a normal subgroup of $(B,\circ)$;
2. $I \cap G$ is a normal subgroup of $(G,+)$;
3. $\rho_b(n) \in I$, for all $b \in B$ and $n \in I \cap G$;
4. $\lambda_g(e) \in I$, for all $g \in G$ and $e \in I \cap E$.

Example 1.3.2. Clearly, B and $\{0\}$ are ideals of B that we call *trivial ideals* of B.

Note that if the additive structure of a left semi-brace is a group, i.e., B is a skew left brace, then the Definition 1.3.1 is equivalent to that of ideal of a left brace introduced in [19].

Definition 1.3.3 (Definition 2.2 in [19])**.** Let B be a skew left brace a normal subgroup of $(B,\circ)$ is said an *ideal* of B if $I + a = a + I$ and $\lambda_a(I) \subseteq I$, for all $a \in B$.

Proposition 1.3.4. *Let B be a skew left brace and $I \subseteq B$. The subset I satisfies the Definition 1.3.1 if and only if I satisfies the Definition 1.3.3.*

Proof. Let I be a subset of B and suppose that I satisfies the Definition 1.3.1. Then I is a normal subgroup of $(B,\circ)$. Moreover, let $a \in B$ and $x \in I$. Then, $y = a^- + x - a^- \in I$ and so

$$\lambda_a(x) = a \circ (a^- + x) = a \circ (y + a^-) = (\rho_{a^-}(y^-))^- \in I,$$

i.e., $\lambda_a(I) \subseteq I$. Hence I is an ideal of the skew left brace B, as in Definition 1.3.3. Conversely, if I is an ideal of B seen as skew left brace, then I is a normal subgroup of $(B,\circ)$ and, clearly, $N = I \cap G = I$ and by hypothesis I is an normal subgroup of the additive group of B. Moreover, let $b \in B$ and $n \in I$. Then, there exists $m \in I$ such that $n^- + b = b + m$ and so

$$\rho_b(n) = (n^- + b)^- \circ b = (b^- \circ (b + m))^- = (0 - b^- + b^- \circ m)^- = (\lambda_{b^-}(m))^- \in I.$$

Finally, let $b \in B$. Thus $\lambda_b(0) = -b + b \circ 0 = 0 \in I$. Therefore, I satisfies Definition 1.3.1. □

Proposition 1.3.5. *Let B be a left semi-brace and I an ideal of B. Then*

$$\rho_b(I) \subseteq I,$$

for every $b \in B$.

Proof. If $x = n + e$ with $n \in G \cap I$ and $e \in E \cap I$ then $x = e \circ (\rho_e(n^-))^-$ and

$$\begin{aligned}\rho_b(x) &= \left(\left(e \circ (\rho_e(n^-))^-\right)^- + b\right)^- \circ b = (\rho_e(n^-) \circ e^- + b)^- \circ b \\ &= \left(\rho_e(n^-) + \lambda_{\rho_e(n^-)}(e^-) + b\right)^- \circ b = (\rho_e(n^-) + b)^- \circ b \\ &= \rho_b\left((\rho_e(n^-))^-\right) \in I\end{aligned}$$

by 3. in Definition 1.3.1. □

There is a difference respect to ideals of a skew left brace. In fact, an ideal of a skew left brace B is λ_a-invariant, for every $a \in B$, this does not happen in a left semi-brace, as the following example shows.

Example 1.3.6. Let B be a trivial left semi-brace where the multiplicative group is not simple and I is a non-trivial normal subgroup of $(B, \circ)$. Then I is an ideal of B. In fact, by hypothesis I is a normal subgroup of $(B, \circ)$. Moreover $N := I \cap G = \{0\}$ and so it is a normal subgroup of $(G, +)$. Now, let $b \in B$; then by 5. in Proposition 1.2.11, $\rho_b(0) = 0 \in N$. Finally, let $e \in I$, $\lambda_0(e) = 0 \circ (0 + e) = e \in I$. Let $a \in B \setminus I$. Although $0 \in I$, we have

$$\lambda_a(0) = a \circ (a^- + 0) = a \circ 0 = a \notin I.$$

Proposition 1.3.7. *Let B be a left semi-brace, I an ideal of B and $\sim_I$ the relation on B given by*

$$\forall\, x, y \in B, \quad x \sim_I y \iff y^- \circ x \in I.$$

Then $\sim_I$ is a congruence of B.

Proof. First we note that $\sim_I$ is the well-known congruence of the group $(B, \circ)$. Moreover, $N := I \cap G$ is a normal subgroup of $(G, \circ)$ and $F := I \cap E$ is a normal

subgroup of $(E, \circ)$. Now, in order to prove that $\sim_I$ is also a congruence of $(B, +)$, we notice that

$$(g_2 + e_2)^- \circ (g_1 + e_1) = (g_2 + e_2)^- \circ (g_1 + e_2) + (g_2 + e_2)^- \circ (g_2 + e_1) \tag{1.6}$$

with $(g_2 + e_2)^- \circ (g_1 + e_2) \in G$ and $(g_2 + e_2)^- \circ (g_2 + e_1) \in E$. In fact,

$$(g_2 + e_2)^- \circ (g_1 + e_2) = (g_2 + e_2)^- \circ e_2 \circ e_2^- \circ (g_1 + e_2) = \rho_{e_2}(g_2^-) \circ (\rho_{e_2}(g_1^-))^- \in G$$

by 2. in Proposition 1.2.11. Moreover,

$$(g_2 + e_2)^- \circ (g_2 + e_1) = \lambda_{(g_2+e_2)^-}(e_1) \in E$$

by 1. in Proposition 1.2.11. Now, we can prove that

$$g_1 + e_1 \sim_I g_2 + e_2 \iff g_1 \sim_I g_2 \quad \text{and} \quad e_1 \sim_I e_2, \tag{1.7}$$

for all $g_1, g_2 \in G$ and $e_1, e_2 \in E$. If $g_1 + e_1 \sim_I g_2 + e_2$ then $(g_2 + e_2)^- \circ (g_1 + e_1) \in I$ and by (1.6) we have

$$(g_2 + e_2)^- \circ (g_1 + e_2) \in N \tag{1.8}$$
$$(g_2 + e_2)^- \circ (g_2 + e_1) \in F \tag{1.9}$$

Note that

$$\begin{aligned}
(g_2 + e_2)^- \circ (g_1 + e_2) &= (g_2 + e_2)^- \circ g_1 + (g_2 + e_2)^- \circ (g_2 + e_2 + e_2) \\
&= (g_2 + e_2)^- \circ g_1 + (g_2 + e_2)^- \circ (g_2 + e_2) = (g_2 + e_2)^- \circ g_1 + 0 \\
&= (g_1^- \circ (g_2 + e_2))^- + 0 = (g_1^- \circ g_2 + g_1^- \circ (g_1 + e_2))^- + 0 \\
&= \rho_{g_1^- \circ (g_1+e_2)}(g_2^- \circ g_1) \circ (g_1^- \circ (g_1 + e_2))^- + 0 \\
&= \rho_{g_1^- \circ (g_1+e_2)}(g_2^- \circ g_1) + \lambda_{\rho_{g_1^- \circ (g_1+e_2)}(g_2^- \circ g_1)}((g_1^- \circ (g_1 + e_2))^-) + 0 \\
&= \rho_{g_1^- \circ (g_1+e_2)}(g_2^- \circ g_1) + 0 = \rho_{g_1^- \circ (g_1+e_2)}(g_2^- \circ g_1).
\end{aligned}$$

By 3. in Definition 1.3.1 and (1.8), we have that

$$g_2^- \circ g_1 = \rho_0(g_2^- \circ g_1) = \rho_{(g_1^- \circ (g_1+e_2))^-}(\rho_{g_1^- \circ (g_1+e_2)}(g_1^- \circ g_2)) \in I,$$

i.e., $g_1 \sim_I g_2$.
Furthermore,

$$\begin{aligned}
(g_2 + e_2)^- \circ (g_2 + e_1) &= \lambda_{(g_2+e_2)^-}(e_1) = \lambda_{\rho_{e_2}(g_2^-) \circ e_2^-}(e_1) \\
&= \lambda_{\rho_{e_2}(g_2^-)}(\lambda_{e_2^-}(e_1)) = \lambda_{\rho_{e_2}(g_2^-)}(e_2^- \circ e_1).
\end{aligned}$$

By (1.9) and 4. in Definition 1.3.1, we have that

$$e_2^- \circ e_1 = \lambda_{(\rho_{e_2}(g_2^-))^-}(\lambda_{\rho_{e_2}(g_2^-)}(e_2^- \circ e_1)) \in I,$$

i.e., $e_1 \sim_I e_2$.
Conversely,

$$(g_2 + e_2)^- \circ (g_1 + e_1) = \rho_{g_1^- \circ (g_1 + e_2)}(g_2^- \circ g_1) + \lambda_{(g_2+e_2)^-}(e_1) \in N + F = I$$

by 3. and 4. in Definition 1.3.1. Therefore, $g_1 + e_1 \sim_I g_2 + e_2$.
Now, let $g_1, g_2 \in G$ and $e_1, e_2 \in E$ such that $g_1 \sim_I g_2$ and $e_1 \sim_I e_2$. By (1.7) we have $g_1 + e_1 \sim_I g_2 + e_2$.
Let g_1, g_2, g_3, g_4 be in G and e_1, e_2, e_3, e_4 be in E such that $g_1 + e_1 \sim_I g_2 + e_2$ and $g_3 + e_3 \sim_I g_4 + e_4$. Note that $g_1 + e_1 + g_3 + e_3 = g_1 + g_3 + e_3$ and $g_2 + e_2 + g_4 + e_4 = g_2 + g_4 + e_4$. Therefore, it suffices to prove that $g_1 + g_3 \sim_I g_2 + g_4$. Observe that N is an ideal of the skew left brace $(G, +, \circ)$ and that if $h, k \in G$ then $h \sim_I k$ if and only if $h \sim_N k$. Moreover, $\sim_N$ is a congruence of G respect to the sum. So $g_1 + g_3 \sim_N g_2 + g_4$, i.e., $g_1 + g_3 \sim_I g_2 + g_4$. Therefore $\sim_I$ is a congruence of B respect to the sum. □

Definition 1.3.8. Let B_1 and B_2 be left semi-braces. A map $\varphi : B_1 \to B_2$ is a *homomorphism* from B_1 into B_2 if φ is a homomorphism from the group $(B_1, \circ)$ into the group $(B_2, \circ)$ and

$$\varphi(a + b) = \varphi(a) + \varphi(b),$$

for all $a, b \in B_1$.

First, note that if B is a left semi-brace and I is an ideal, then the quotient structure of B respect to the relation $\sim_I$ is a right group respect to the sum. In fact, if $b, b_1, b_2 \in B$ such that $b + b_1 \sim_I b + b_2$ then, by (1.7) in previous proposition, $b_1 \sim_I b_2$. Moreover, if $b_1 = g_1 + e_1$ and $b_2 = g_2 + e_2$ then, setting $x := -g_1 + g_2 + e_2$, we have $b_1 + x \sim_I b_2$. Therefore, B/I is a left semi-brace and the canonical epimorphism $\pi : B \to B/I$, $b \mapsto I \circ b$ is an epimorphism of left semi-braces. Conversely, let B, C be left semi-braces and $\varphi : B \to C$ a homomorphism of left semi-braces. Then $\ker \varphi := \{b \mid b \in B\ \varphi(b) = 0\}$ is an ideal of B. In fact, clearly $\ker f$ is a normal subgroup of the multiplicative group of B. Moreover, let E be the set of idempotents of B and $G = B + 0$, $N := \ker \varphi \cap G$ and $F := \ker \varphi \cap E$. Then N is a normal subgroup of the group $(G, +)$. In fact let $g \in G$ and $n \in N$. It follows $-g + n + g \in G$ and

$$\varphi(-g + n + g) = -\varphi(g) + \varphi(n) + \varphi(g) = -\varphi(g) + 0 + \varphi(g) = 0,$$

i.e., $-g+n+g \in N$. Let, now, $b \in B$ and $n \in N$. Then $\rho_b(n) \in \ker\varphi$. In fact,

$$\varphi(\rho_b(n)) = \left(\varphi(n)^- + \varphi(b)\right)^- \circ \varphi(b) = (0+\varphi(b))^- \circ \varphi(b) = \varphi(b)^- \circ \varphi(b) = 0,$$

i.e., $\rho_b(n) \in \ker\varphi$. Finally, let $g \in G$ and $e \in F$. Thus,

$$\varphi(\lambda_g(e)) = \varphi(g \circ (g^- + e)) = \varphi(g) \circ \left(\varphi(g)^- + \varphi(e)\right) = \varphi(g) \circ \varphi(g)^- = 0,$$

i.e., $\lambda_g(e) \in \ker\varphi$.
Therefore, we may conclude that a subset I of a left semi-brace B is an ideal of B if and only if I is the kernel of a certain homomorphism of left semi-braces.

Example 1.3.9. Let $B_1 := H + E_1, B_2 := N + E_2$ be left semi-braces, B a semidirect product of B_1 and B_2 and J an ideal of B_2, as in Corollary 2.1.5. Then $I := \{(h+e, t+j) \mid h+e \in B_1, t+j \in J\}$ is an ideal of B. In fact, let $(h+e, t+j) \in I$ and $(h_1+e_1, n_1+f_1) \in B$. Since J is a normal subgroup of B_2, then I a normal subgroup of $(B, \circ)$. In fact,

$$\begin{aligned}
&(h_1+e_1, n_1+f_1)^- \circ (h+e, t+j) \circ (h_1+e_1, n_1+f_1)\\
&= \left({}^{(n_1+f_1)^-}(h_1+e_1)^-, (n_1+f_1)^-\right) \circ \left((h+e) \circ {}^{t+j}(h_1+e_1), (t+j) \circ (n_1+f_1)\right)\\
&= \Big({}^{(n_1+f_1)^-}(h_1+e_1)^- \circ {}^{(n_1+f_1)^-}\left((h+e) \circ {}^{t+j}(h_1+e_1)\right),\\
&\quad (n_1+f_1)^- \circ (t+j) \circ (n_1+f_1)\Big) \in H.
\end{aligned}$$

Moreover, obviously $I \cap G$ is a normal subgroup of $(G, +)$. Finally,

$$\rho_{(h_1+e_1, n_1+f_1)}(h,t) = \left({}^{(n_1+f_1)^-}\left(\left({}^{t^-}h^- + h_1 + e_1\right)^- \circ (h_1+e_1)\right), \rho_{n_1+f_1}(t)\right) \in I$$

for all $(h_1+e_1, n_1+f_1) \in B$ and $(h,t) \in J$, and

$$\lambda_{(h,n)}(e,j) = \left(h \circ \left(h^- + {}^n e\right), \lambda_n(j)\right) \in I$$

for all $(h,n) \in G$ and $(e,j) \in E \cap I$, since J is an ideal of B_2. So J is an ideal of B. In particular, when $J = \{0\}$ the projection $\overline{B}_1 := \{(h+e, 0) \mid h+e \in B_1\}$ is an ideal of B. Finally the quotient structure of B with respect to the ideal $\overline{B}_1$ is isomorphic to the left semi-brace B_2.

Now we introduce a particular kind of ideal, the socle. This ideal allows us to obtain a new left semi-brace from a left semi-brace.

Definition 1.3.10. Let B be a left semi-brace. We call the set given by

$$\mathrm{Soc}(B) := \{a \mid a \in B \quad \lambda_a = \lambda_0,\ \rho_a = \rho_0\}$$

the *socle* of the left semi-brace B.

Proposition 1.3.11. *Let B be a left semi-brace. Then $\mathrm{Soc}(B)$ is an ideal of B.*

Proof. First, $\mathrm{Soc}(B)$ is a subgroup of the multiplicative group $(B,\circ)$. In fact, clearly $0 \in \mathrm{Soc}(B)$. If x is in $\mathrm{Soc}(B)$, then

$$\begin{aligned}\lambda_{x^-} &= \lambda_x^{-1} = \lambda_0^{-1} = \lambda_0,\\ \rho_{x^-} &= \rho_{0\circ x^-} = \rho_{x^-}\rho_0 = \rho_{x^-}\rho_x = \rho_{x\circ x^-} = \rho_0,\end{aligned}$$

i.e., $x^- \in \mathrm{Soc}(B)$. Now, if x, y are in $\mathrm{Soc}(B)$, then

$$\begin{aligned}\lambda_{x\circ y} &= \lambda_x\lambda_y = \lambda_0\lambda_0 = \lambda_{0\circ 0} = \lambda_0,\\ \rho_{x\circ y} &= \rho_y\rho_x = \rho_0\rho_0 = \rho_{0\circ 0} = \rho_0,\end{aligned}$$

i.e., $x\circ y \in \mathrm{Soc}(B)$. Let $a, b \in B$ and $x \in \mathrm{Soc}(B)$. Then,

$$\begin{aligned}\lambda_{b^-\circ x\circ b}(a) &= \lambda_{b^-}(\lambda_x(\lambda_b(a))) = \lambda_{b^-}(\lambda_0(\lambda_b(a))) = \lambda_{b^-\circ b}(a) = \lambda_0(a),\\ \rho_{b^-\circ x\circ b}(a) &= \rho_b(\rho_x(\rho_{b^-}(a))) = \rho_b(\rho_0(\rho_{b^-}(a))) = \rho_{b\circ b^-}(a) = \rho_0(a),\end{aligned}$$

i.e., $\mathrm{Soc}(B)$ is a normal subgroup of the multiplicative group of B. Moreover, if $x \in \mathrm{Soc}(B)$, then $x^-\circ(x+b) = \lambda_{x^-}(b) = \lambda_0(b) = b$, for every $b \in B$, i.e., $x+b = x\circ b$, for every $b \in B$. Note that

$$\lambda_x(0) = x\circ(x^- + 0) = x\circ(x^-\circ 0) = 0,$$

and so by 4. in Proposition 1.2.11, $x \in G$. Therefore $\mathrm{Soc}(B)$ is a subgroup of $(G,+)$. Further, note that, if x is in $\mathrm{Soc}(B)$, then $(b+x)^-\circ x = \rho_x(b^-) = \rho_0(b^-) = (b+0)^-$, for every $b \in B$, i.e., $b + x = x\circ(b+0)$, for every $b \in B$. Now, if x is in $\mathrm{Soc}(B)$ and g is in G, then

$$\begin{aligned}-g+x+g &= x\circ(-g+0)+g = x\circ(-g)+g = x+(-g)+g = x+0\\ &= x \in \mathrm{Soc}(B),\end{aligned}$$

thus $\mathrm{Soc}(B)$ is a normal subgroup of $(G,+)$. Let, next, $b \in B$ and $n \in \mathrm{Soc}(B)$. Then

$$\rho_b(n) = (n^- + b)^-\circ b = (n^-\circ b)^-\circ b = b^-\circ n\circ b \in \mathrm{Soc}(B).$$

Finally, let $g \in G$. Then, by 4. in Proposition 1.2.11, $\lambda_g(0) = 0$ is in $\mathrm{Soc}(B)$. Thus $\mathrm{Soc}(B)$ is an ideal of B. □

Even if the additive structure of every ideal of a left semi-brace is a semigroup, the additive structure of the socle of B is a group.

Proposition 1.3.12. *Let B be a left semi-brace. Then $(\mathrm{Soc}(B), +)$ is a subgroup of the semigroup $(B, +)$. In particular, $-a = a^-$, for every $a \in B$.*

Proof. Let a be in $\mathrm{Soc}(B)$. Note that

$$a^- + a = a^- \circ \lambda_a(a) = a^- \circ \lambda_0(a) = a^- \circ a = 0$$

and since a^- is in $\mathrm{Soc}(B)$, then $\mathrm{Soc}(B)$ is a subgroup of $(B, +)$ where the opposite of a is a^-. □

The following proposition describes the socle of a left semi-brace only in terms of the two operations of the left semi-brace.

Proposition 1.3.13. *Let B be a left semi-brace, $G = B + 0$. Then*

$$\mathrm{Soc}(B) = \{a \mid a \in G,\ \forall\, b \in B \quad a + b = a \circ b,\ -a + b + a = b + 0\}.$$

Proof. For convenience, we set

$$\mathcal{S} := \{a \mid a \in G,\ \forall\, b \in B \quad a + b = a \circ b,\ -a + b + a = b + 0\}.$$

Let $a \in \mathcal{S}$ and $b \in B$. Then,

$$\lambda_a(b) = a \circ (a^- + b) = a + a^- + b = (a \circ a^-) + b = 0 + b = \lambda_0(b).$$

Moreover,

$$\begin{aligned}\rho_a(b) &= (b^- + a)^- \circ a = (a - a + b^- + a)^- \circ a = (a + (b^- + 0))^- \circ a \\ &= (a \circ (b^- + 0))^- \circ a = (b^- + 0)^- \circ a^- \circ a = (b^- + 0)^- = \rho_0(b).\end{aligned}$$

Conversely, let a be in $\mathrm{Soc}(B)$. By previous proposition $a \in G$. Moreover, if $b \in B$, then $a^- \circ (a + b) = \lambda_{a^-}(b) = \lambda_0(b) = b$, since $a^- \in \mathrm{Soc}(B)$, and so $a + b = a \circ b$. Finally, by Proposition 1.3.12,

$$-a + b + a = a^- + b + a = a^- \circ (b + a) = (\rho_a(b^-))^- = (\rho_0(b^-))^- = b + 0.$$

Therefore, a is in $\mathcal{S}$. □

Remark 1.3.14. By Proposition 1.3.12, the socle of a left semi-brace is a skew left brace, in particular, by Proposition 1.3.13, it is a zero skew left brace.

If B is a skew left brace then Definition 1.3.1 coincides with the definition of socle in [19].

Definition 1.3.15 (Definition 2.4 in [19])**.** Let B be a skew left brace, the *socle* is the set

$$S = \{a \mid a \in B,\ \forall\, b \in B \quad a \circ b = a + b,\ b + b \circ a = b \circ a + b\}\,.$$

Proposition 1.3.16. *Let B be a skew left brace. Then the Definition 1.3.10 and Definition 1.3.15 coincide.*

Proof. By Proposition 1.3.13, if B is seen as a left semi-brace

$$\mathrm{Soc}(B) = \{a \mid a \in B,\ \forall\, b \in B \quad a \circ b = a + b,\ b + a = a + b\}\,.$$

If $a \in S$ and $b \in B$, then

$$\begin{aligned} b + a &= b \circ b^- \circ (b + a) = b \circ \left(b^- \circ b - b^- + b^- \circ a\right) \\ &= b \circ \left(-b^- + b^- \circ a + 0\right) = b \circ \left(-b^- + b^- \circ a + b^- - b^-\right) \\ &= b \circ \left(-b^- + b^- + b^- \circ a - b^-\right) = b \circ \left(b^- \circ a - b^-\right) \\ &= b \circ \left(b^- \circ a - b^- + 0\right) = b \circ \left(b^- \circ a - b^- + b^- \circ b\right) \\ &= b \circ b^- \circ (a + b) = a + b, \end{aligned}$$

i.e., $a \in \mathrm{Soc}(B)$. Conversely, let $a \in \mathrm{Soc}(B)$ and $b \in B$. Then,

$$\begin{aligned} b \circ a + b &= b + (-b + b \circ a) + b = b + (0 - b + b \circ a) + b \\ &= b + (b \circ b^- - b + b \circ a) + b = b + b \circ (b^- + a) + b \\ &= b + b \circ (a + b^-) + b = b + b \circ a - b + b \circ b^- + b \\ &= b + b \circ a - b + b = b + b \circ a, \end{aligned}$$

i.e., a is in S. □

Example 1.3.17. Let B be the left semi-brace in Example 1.1.4. Then

$$\mathrm{Soc}(B) = Z(B) \cap \mathrm{Im}\, f,$$

where $Z(B)$ is the centre of the group $(B, \circ)$. In fact, let a be in $\mathrm{Soc}(B)$. By Proposition 1.3.13, a lies in $\mathrm{Im}\, f$, then there exists $c \in B$ such that $a = f(c)$. Moreover $a \circ b = f(c) \circ b = f(c) + b = b \circ f^2(c) = b \circ f(c) = b \circ a$, for every $b \in B$. So $a \in Z(B)$. Conversely, let $a \in Z(B) \cap \mathrm{Im}\, f$. Then there exists $c \in B$ such that $a = f(c)$, $a + b = f(c) + b = b \circ f(c) = b \circ a = a \circ b$ and $-a + b + a = a^- + b + a = f(c)^- + b + f(c) = b \circ f(c)^- + f(c) = f(c) \circ f(b) \circ f(c)^- = f(b) \circ f(c) \circ f(c)^- = f(b) = b + 0$, for every $b \in B$. Therefore a is in $\mathrm{Soc}(B)$.

Definition 1.3.18. Let B be a left semi-brace, E the set of idempotents of $(B,+)$, 0 the identity of $(B,\circ)$ and $G := B+0$. The *annihilator of* B is the set

$$\operatorname{Ann}(B) := \{a \mid a \in G,\ \forall b \in B \quad a+b = a\circ b = b\circ a,\ -a+b+a = b+0\}.$$

Remark 1.3.19. Let B be a left semi-brace, E the set of idempotents of $(B,+)$, 0 the identity of $(B,\circ)$ and $G := B+0$. Then

$$\operatorname{Ann}(B) = \operatorname{Soc}(B) \cap \mathrm{Z}(B),$$

where $\mathrm{Z}(B)$ is the centre of the group $(B,\circ)$.

Proposition 1.3.20. *Let B be a left semi-brace, E the set of idempotents of $(B,+)$, 0 the identity of $(B,\circ)$ and $G := B+0$. Then $\operatorname{Ann}(B)$ is an ideal of B.*

Proof. By Remark 1.3.19, $\operatorname{Ann}(B) = \operatorname{Soc}(B) \cap \mathrm{Z}(B)$ and so $\operatorname{Ann}(B)$ is a normal subgroup of $(B,\circ)$, since both $\operatorname{Soc}(B)$ and $\mathrm{Z}(B)$ are normal subgroups. Further $\operatorname{Ann}(B) \cap G = \operatorname{Ann}(B)$ is a normal subgroup of $(G,+)$. In fact, if $a_1, a_2 \in \operatorname{Ann}(B)$, then $a_1 - a_2 = a_1 \circ a_2^- \in \operatorname{Ann}(B)$, moreover if $a \in \operatorname{Ann}(B)$ and $g \in G$, then

$$-g+a+g = a-g+0+g = a \in \operatorname{Ann}(B).$$

Finally, if $a \in \operatorname{Ann}(B)$ and $b \in B$, then

$$\rho_b(a) = \left(a^- + b\right)^- \circ b = \left(a^- \circ b\right)^- \circ b = b^- \circ a \circ b = a \circ b^- \circ b = a \in \operatorname{Ann}(B).$$

Therefore, by Definition 1.3.1, $\operatorname{Ann}(B)$ is an ideal of B, since $\operatorname{Ann}(B) \cap E = \{0\}$ and condition $4.$ is trivially satisfied. □

1.3.1 Extensions of skew left braces

This construction is a generalization to skew left braces of the construction of left braces introduced by Bachiller in [2].

Recall that if $(B,+)$ is a group and $(I,+)$ is an abelian group, then a map $\tau : B\times B \to I$ is a *2-cocycle from* $(B,+)$ *with values in* $(I,+)$ if the following conditions hold:

1. $\tau(b_1+b_2,b_3) + \tau(b_1,b_2) = \tau(b_1,b_2+b_3) + \tau(b_2,b_3)$, for all $b_1,b_2,b_3 \in B$;
2. $\tau(b,0) = \tau(0,b) = 0$, for every $b \in B$.

Moreover if $(B,\circ)$ is a group, $(I,+)$ is an abelian group and $\sigma : B \to \operatorname{Aut}(I)$ is a right action, then a map $\theta : B \times B \to I$ is a *2-cocycle from* $(B,\circ)$ *with values in* $(I,+)$ *respect to* σ if the following conditions hold

1. $\sigma_{b_3}(\theta(b_1,b_2))+\theta(b_1\circ b_2,b_3)=\theta(b_1,b_2\circ b_3)+\theta(b_2,b_3)$, for all $b_1,b_2,b_3\in B$;

2. $\theta(b,0)=\theta(0,b)=0$, for every $b\in B$.

Theorem 1.3.21. *Let B be a skew left brace, I a zero left brace, $\sigma: B\to \mathrm{Aut}(I)$ a right action from $(B,\circ)$ into the group of automorphisms of $(I,+)$, $\nu: B\to \mathrm{Aut}(I)$ a left action from $(B,\circ)$ into the group of automorphisms of $(I,+)$, $\tau: B\times B\to I$ a 2-cocycle from $(B,+)$ with values in $(I,+)$ and $\theta: B\times B\to I$ a 2-cocycle from $(B,\circ)$ with values in $(I,+)$ respect to σ. If the following conditions hold:*

$$\nu_{b_1+b_2}(\sigma_{b_1+b_2}(i))+i=\nu_{b_1}(\sigma_{b_1}(i))+\nu_{b_2}(\sigma_{b_2}(i)); \tag{1.10}$$

$$\begin{aligned}&\nu_{b_1\circ(b_2+b_3)}(\theta(b_1,b_2+b_3))+\nu_{b_1}(\tau(b_2,b_3))=\nu_{b_1\circ b_2}(\theta(b_1,b_2))\\&+\nu_{b_1\circ b_3}(\theta(b_1,b_3))-\tau(b_1,-b_1+b_1\circ b_3)+\tau(b_1\circ b_2,-b_1+b_1\circ b_3),\end{aligned} \tag{1.11}$$

for all $b,b_1,b_2,b_3\in B$ and $i\in I$, then $B\times I$ is a skew left brace with the sum

$$(b_1,i_1)+(b_2,i_2):=(b_1+b_2,i_1+i_2+\tau(b_1,b_2))$$

and the product

$$(b_1,i_1)\circ(b_2,i_2):=\left(b_1\circ b_2,\ \nu_{b_1\circ b_2}\left(\sigma_{b_2}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_2)+\nu_{b_1\circ b_2}(\theta(b_1,b_2))\right),$$

for all $(b_1,i_1),(b_2,i_2)\in B\times I$. Moreover, $\{0\}\times I$ is an ideal of $B\times I$ contained in the centre of $(B\times I,+)$ such that $(B\times I)/(\{0\}\times I)\cong B$ and $\{0\}\times I$ is isomorphic to I.

Conversely, if B is a skew left brace, I is an ideal of B contained in the centre of $(B,+)$ such that I is a trivial left brace and $\bar{B}:=B/I$, then B is isomorphic to a skew left brace of the form described above, constructed from $\bar{B}$ and I.

Proof. Since τ is a 2-cocycle, then $(B\times I)$ is a group with identity $(0,0)$ and inverse $(-b,-i-\tau(b,-b))$, for every $(b,i)\in B\times I$. Further $(B\times I,\circ)$ is a group with identity $(0,0)$ and inverse $(b^-,-\nu_{b^-}(\sigma_{b^-}(\nu_{b^-}(i)))-\nu_{b^-}(\theta(b,b^-)))$, for every $(b,i)\in B\times I$. In fact, if $b\in B$ and $i\in I$, then

$$(0,0)\circ(b,i)=(0\circ b,\ \nu_{0\circ b}(\sigma_b(\nu_0(0)))+\nu_0(i)+\nu_{0\circ b}(\theta(0,b)))=(b,i),$$

$$(b,i)\circ(0,0)=(b\circ 0,\ \nu_{b\circ 0}(\sigma_0(\nu_{b^-}(i)))+\nu_b(0)+\nu_{b\circ 0}(\theta(b,0)))=(b,i).$$

and

$$
\begin{aligned}
&(b,i)\circ\left(b^{-},-\nu_{b^-}\left(\sigma_{b^-}\left(\nu_{b^-}\left(i\right)\right)\right)-\nu_{b^-}\left(\theta\left(b,b^{-}\right)\right)\right)\\
&=\left(0,\ \nu_0\left(\sigma_{b^-}\left(\nu_{b^-}\left(i\right)\right)\right)+\nu_b\left(-\nu_{b^-}\left(\sigma_{b^-}\left(\nu_{b^-}\left(i\right)\right)\right)-\nu_{b^-}\left(\theta\left(b,b^{-}\right)\right)\right)+\nu_0\left(\theta\left(b,b^{-}\right)\right)\right)\\
&=\left(0,\ \sigma_{b^-}\left(\nu_{b^-}\left(i\right)\right)-\sigma_{b^-}\left(\nu_{b^-}\left(i\right)\right)-\theta\left(b,b^{-}\right)+\theta\left(b,b^{-}\right)\right)=(0,0)\,.
\end{aligned}
$$

If $b_1,b_2,b_3\in B$ and $i_1,i_2,i_3\in I$, then

$$
\begin{aligned}
&(b_1,i_1)\circ\left((b_2,i_2)\circ(b_3,i_3)\right)\\
&=(b_1,i_1)\circ\left(b_2\circ b_3,\ \nu_{b_2\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_2^-}\left(i_2\right)\right)\right)+\nu_{b_2}\left(i_3\right)+\nu_{b_2\circ b_3}\left(\theta\left(b_2,b_3\right)\right)\right)\\
&=\left(b_1\circ b_2\circ b_3,\ \nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_2\circ b_3}\left(\nu_{b_1^-}\left(i_1\right)\right)\right)+\nu_{b_1}\left(\nu_{b_2\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_2^-}\left(i_2\right)\right)\right)+\nu_{b_2}\left(i_3\right)\right.\right.\\
&\left.\left.+\nu_{b_2\circ b_3}\left(\theta\left(b_2,b_3\right)\right)\right)+\nu_{b_1\circ b_2\circ b_3}\left(\theta\left(b_1,b_2\circ b_3\right)\right)\right)\\
&=\left(b_1\circ b_2\circ b_3,\ \nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_2\circ b_3}\left(\nu_{b_1^-}\left(i_1\right)\right)\right)+\nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_2^-}\left(i_2\right)\right)\right)+\nu_{b_1\circ b_2}\left(i_3\right)\right.\\
&\left.+\nu_{b_1\circ b_2\circ b_3}\left(\theta\left(b_2,b_3\right)\right)+\nu_{b_1\circ b_2\circ b_3}\left(\theta\left(b_1,b_2\circ b_3\right)\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left((b_1,i_1)\circ(b_2,i_2)\right)\circ(b_3,i_3)\\
&=\left(b_1\circ b_2,\ \nu_{b_1\circ b_2}\left(\sigma_{b_2}\left(\nu_{b_1^-}\left(i_1\right)\right)\right)+\nu_{b_1}\left(i_2\right)+\nu_{b_1\circ b_2}\left(\theta\left(b_1,b_2\right)\right)\right)\circ(b_3,i_3)\\
&=\left(b_1\circ b_2\circ b_3,\ \nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_3}\left(\nu_{(b_1\circ b_2)^-}\left(\nu_{b_1\circ b_2}\left(\sigma_{b_2}\left(\nu_{b_1^-}\left(i_1\right)\right)\right)+\nu_{b_1}\left(i_2\right)\right.\right.\right.\right.\\
&\left.\left.\left.\left.+\nu_{b_1\circ b_2}\left(\theta\left(b_1,b_2\right)\right)\right)\right)\right)+\nu_{b_1\circ b_2}\left(i_3\right)+\nu_{b_1\circ b_2\circ b_3}\left(\theta\left(b_1\circ b_2,b_3\right)\right)\right)\\
&=\left(b_1\circ b_2\circ b_3,\ \nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_2\circ b_3}\left(\nu_{b_1^-}\left(i_1\right)\right)\right)+\nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_2^-}\left(i_2\right)\right)\right)\right.\\
&\left.+\nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_3}\left(\theta\left(b_1,b_2\right)\right)\right)+\nu_{b_1\circ b_2}\left(i_3\right)+\nu_{b_1\circ b_2\circ b_3}\left(\theta\left(b_1\circ b_2,b_3\right)\right)\right)\\
&=\left(b_1\circ b_2\circ b_3,\ \nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_2\circ b_3}\left(\nu_{b_1^-}\left(i_1\right)\right)\right)+\nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_2^-}\left(i_2\right)\right)\right)\right.\\
&\left.+\nu_{b_1\circ b_2}\left(i_3\right)+\nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_3}\left(\theta\left(b_1,b_2\right)\right)+\theta\left(b_1\circ b_2,b_3\right)\right)\right)\\
&=\left(b_1\circ b_2\circ b_3,\ \nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_2\circ b_3}\left(\nu_{b_1^-}\left(i_1\right)\right)\right)+\nu_{b_1\circ b_2\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_2^-}\left(i_2\right)\right)\right)\right.\\
&\left.+\nu_{b_1\circ b_2}\left(i_3\right)+\nu_{b_1\circ b_2\circ b_3}\left(\theta\left(b_2,b_3\right)+\theta\left(b_1,b_2\circ b_3\right)\right)\right)=(b_1,i_1)\circ\left((b_2,i_2)\circ(b_3,i_3)\right),
\end{aligned}
$$

i.e., the associativity of $\circ$ holds. If $b_1,b_2,b_3\in B$ and $i_1,i_2,i_3\in I$, then

$$
\begin{aligned}
&(b_1,i_1)\circ\left((b_2,i_2)+(b_3,i_3)\right)=(b_1,i_1)\circ\left(b_2+b_3,i_2+i_3+\tau\left(b_2,b_3\right)\right)\\
&=\left(b_1\circ\left(b_2+b_3\right),\ \nu_{b_1\circ(b_2+b_3)}\left(\sigma_{b_2+b_3}\left(\nu_{b_1^-}\left(i_1\right)\right)\right)+\nu_{b_1}\left(i_2+i_3+\tau\left(b_2,b_3\right)\right)\right.\\
&\left.+\nu_{b_1\circ(b_2+b_3)}\left(\theta\left(b_1,b_2+b_3\right)\right)\right)
\end{aligned}
$$

moreover

$$
\begin{aligned}
&-(b_1,i_1)+(b_1,i_1)\circ(b_3,i_3)=(-b_1,-i_1-\tau(b_1,-b_1))\\
&+\left(b_1\circ b_3,\ \nu_{b_1\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_3)+\nu_{b_1\circ b_3}(\theta(b_1,b_3))\right)\\
&=\left(-b_1+b_1\circ b_3,\ -i_1-\tau(b_1,-b_1)+\nu_{b_1\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_3)\right.\\
&\left.+\nu_{b_1\circ b_3}(\theta(b_1,b_3))+\tau(-b_1,b_1\circ b_3)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&(b_1,i_1)\circ(b_2,i_2)-(b_1,i_1)+(b_1,i_1)\circ(b_3,i_3)\\
&=\left(b_1\circ b_2,\ \nu_{b_1\circ b_2}\left(\sigma_{b_2}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_2)+\nu_{b_1\circ b_2}(\theta(b_1,b_2))\right)\\
&+\left(-b_1+b_1\circ b_3,\ -i_1-\tau(b_1,-b_1)+\nu_{b_1\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_3)\right.\\
&\left.+\nu_{b_1\circ b_3}(\theta(b_1,b_3))+\tau(-b_1,b_1\circ b_3)\right)\\
&=\left(b_1\circ(b_2+b_3),\ \nu_{b_1\circ b_2}\left(\sigma_{b_2}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_2)+\nu_{b_1\circ b_2}(\theta(b_1,b_2))\right.\\
&-i_1-\tau(b_1,-b_1)+\nu_{b_1\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_3)+\nu_{b_1\circ b_3}(\theta(b_1,b_3))\\
&\left.+\tau(-b_1,b_1\circ b_3)+\tau(b_1\circ b_2,-b_1+b_1\circ b_3)\right),
\end{aligned}
$$

so the first components are the same for the second ones we may apply conditions (1.10) and (1.11) and we obtain the equality

$$
\begin{aligned}
&\nu_{b_1\circ b_2}\left(\sigma_{b_2}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_2)+\nu_{b_1\circ b_2}(\theta(b_1,b_2))-i_1-\tau(b_1,-b_1)\\
&+\nu_{b_1\circ b_3}\left(\sigma_{b_3}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_3)+\nu_{b_1\circ b_3}(\theta(b_1,b_3))+\tau(-b_1,b_1\circ b_3)\\
&+\tau(b_1\circ b_2,-b_1+b_1\circ b_3)=\nu_{b_1}\left(\nu_{b_2}\left(\sigma_{b_2}\left(\nu_{b_1^-}(i_1)\right)\right)-\nu_{b_1^-}(i_1)\right.\\
&\left.+\nu_{b_3}\left(\sigma_{b_3}\left(\nu_{b_1^-}(i_1)\right)\right)\right)+\nu_{b_1}(i_2+i_3)+\nu_{b_1\circ b_2}(\theta(b_1,b_2))+\nu_{b_1\circ b_3}(\theta(b_1,b_3))\\
&-\tau(b_1,-b_1)+\tau(-b_1,b_1\circ b_3)+\tau(b_1\circ b_2,-b_1+b_1\circ b_3)\\
&=\nu_{b_1\circ(b_2+b_3)}\left(\sigma_{b_2+b_3}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_2+i_3)+\nu_{b_1\circ b_2}(\theta(b_1,b_2))\\
&+\nu_{b_1\circ b_3}(\theta(b_1,b_3))+\tau(0,b_1\circ b_3)-\tau(b_1,-b_1+b_1\circ b_3)+\tau(b_1\circ b_2,-b_1+b_1\circ b_3)\\
&=\nu_{b_1\circ(b_2+b_3)}\left(\sigma_{b_2+b_3}\left(\nu_{b_1^-}(i_1)\right)\right)+\nu_{b_1}(i_2+i_3)+\nu_{b_1}(\tau(b_2,b_3))\\
&+\nu_{b_1\circ(b_2+b_3)}(\theta(b_1,b_2+b_3))
\end{aligned}
$$

hence (1.2) in Definition 1.1.5 holds. Therefore $(B\times I,\circ,+)$ is a skew left brace. Further the map $\varphi: B\times I\to B$ defined by $\varphi(b,i):=b$, for every $(b,i)\in B\times I$ is a surjective homomorphism of skew left braces. Moreover the kernel of φ is

$$\ker\varphi=\{(0,i)\mid i\in I\}=\{0\}\times I.$$

Therefore $(B \times I) / (\{0\} \times I)$ is isomorphic to B. Moreover, if $b \in B$, $i, j \in I$, then

$$\begin{aligned}(0,i)+(b,j) &= (0+b, i+j+\tau(0,b)) = (b, i+j) = (b+0, j+i+\tau(b,0)) \\ &= (b,j)+(0,i),\end{aligned}$$

i.e., $(0,i)$ lies in the centre of $(B \times I, +)$.

On the other hand, let B be a skew left brace, I an ideal of B with zero structure contained in the centre of $(B,+)$. Set $\bar{B} := B/I$. First note that $(I,+)$ is abelian. Consider $\pi : B \to \bar{B}$ the projection map, $s : \bar{B} \to B$ a map such that $s(\bar{0}) = 0$ and $\pi\left(s\left(\bar{b}\right)\right) = \bar{b}$, for every $\bar{b} \in \bar{B}$. Consider the map $\tau : \bar{B} \times \bar{B} \to I$ defined by

$$\tau\left(\bar{b}_1, \bar{b}_2\right) := s\left(\bar{b}_1\right) + s\left(\bar{b}_2\right) - s\left(\bar{b}_1 + \bar{b}_2\right),$$

for all $\bar{b}_1, \bar{b}_2 \in \bar{B}$, the map $\theta : \bar{B} \times \bar{B} \to I$ defined by

$$\theta\left(\bar{b}_1, \bar{b}_2\right) := \left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right),$$

for all $\bar{b}_1, \bar{b}_2 \in \bar{B}$, the map $\sigma : \bar{B} \to \operatorname{Aut}(I)$ defined by

$$\sigma_{\bar{b}} : I \to I, \quad i \mapsto \left(s\left(\bar{b}\right)\right)^{-} \circ i \circ s\left(\bar{b}\right),$$

for every $\bar{b} \in \bar{B}$ and the map $\nu : \bar{B} \to \operatorname{Aut}(I)$ defined by

$$\nu_{\bar{b}} : I \to I, \quad i \mapsto \lambda_{s\left(\bar{b}\right)}(i),$$

for every $\bar{b} \in \bar{B}$. Then $\bar{B}, I, \tau, \theta, \sigma, \nu$ satisfy the first part of this theorem. First, since I is an ideal, then $\left(s\left(\bar{b}\right)\right)^{-} \circ i \circ s\left(\bar{b}\right) \in I$, for all $\bar{b} \in \bar{B}$ and $i \in I$, further if $i \in I$ and $\bar{b}_1, \bar{b}_2 \in \bar{B}$, then

$$\sigma_{\bar{0}}(i) = (s(\bar{0}))^{-} \circ i \circ s(\bar{0}) = 0 \circ i \circ 0 = i = \mathrm{id}_I(i)$$

and

$$\begin{aligned}
&\sigma_{\bar{b}_1}\left(\sigma_{\bar{b}_2}(i)\right) = \left(s\left(\bar{b}_1\right)\right)^{-} \circ \left(\left(s\left(\bar{b}_2\right)\right)^{-} \circ i \circ s\left(\bar{b}_2\right)\right) \circ s\left(\bar{b}_1\right) \\
&= \left(s\left(\bar{b}_2 \circ \bar{b}_1\right)\right)^{-} \circ s\left(\bar{b}_2 \circ \bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) \circ s\left(\bar{b}_1\right)\right)^{-} \circ i \circ s\left(\bar{b}_2\right) \circ s\left(\bar{b}_1\right) \\
&= \left(s\left(\bar{b}_2 \circ \bar{b}_1\right)\right)^{-} \circ \left(s\left(\bar{b}_2 \circ \bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) \circ s\left(\bar{b}_1\right)\right)^{-} + i\right) \circ s\left(\bar{b}_2\right) \circ s\left(\bar{b}_1\right) \\
&= \left(s\left(\bar{b}_2 \circ \bar{b}_1\right)\right)^{-} \circ \left(i + s\left(\bar{b}_2 \circ \bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) \circ s\left(\bar{b}_1\right)\right)^{-}\right) \circ s\left(\bar{b}_2\right) \circ s\left(\bar{b}_1\right) \\
&= \left(s\left(\bar{b}_2 \circ \bar{b}_1\right)\right)^{-} \circ \left(i \circ s\left(\bar{b}_2 \circ \bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) \circ s\left(\bar{b}_1\right)\right)^{-}\right) \circ s\left(\bar{b}_2\right) \circ s\left(\bar{b}_1\right) \\
&= \left(s\left(\bar{b}_2 \circ \bar{b}_1\right)\right)^{-} \circ i \circ s\left(\bar{b}_2 \circ \bar{b}_1\right) = \sigma_{\bar{b}_2 \circ \bar{b}_1}(i),
\end{aligned}$$

since $s\left(\bar{b}_2 \circ \bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) \circ s\left(\bar{b}_1\right)\right)^{-} \in I$ and I has trivial structure. Moreover if $i_1, i_2 \in I$ and $\bar{b} \in \bar{B}$, then

$$\begin{aligned}
&\sigma_{\bar{b}}\left(i_1 + i_2\right) = \left(s\left(\bar{b}\right)\right)^{-} \circ \left(i_1 + i_2\right) \circ s\left(\bar{b}\right) = \left(s\left(\bar{b}\right)\right)^{-} \circ \left(i_1 \circ i_2\right) \circ s\left(\bar{b}\right) \\
&= \left(s\left(\bar{b}\right)\right)^{-} \circ i_1 \circ s\left(\bar{b}\right) \circ \left(s\left(\bar{b}\right)\right)^{-} \circ i_2 \circ s\left(\bar{b}\right) = \sigma_{\bar{b}}\left(i_1\right) \circ \sigma_{\bar{b}}\left(i_2\right) \\
&= \sigma_{\bar{b}}\left(i_1\right) + \sigma_{\bar{b}}\left(i_2\right).
\end{aligned}$$

Hence σ is a right action from $\left(\bar{B}, \circ\right)$ into the group of automorphisms of $(I, +)$. First $\nu_{\bar{b}}\left(i\right) = \lambda_{s\left(\bar{b}\right)}\left(i\right) \in I$, since I is an ideal. Further, if $i \in I$, $\bar{b}_1, \bar{b}_2 \in \bar{B}$, then

$$\nu_{\bar{0}}\left(i\right) = \lambda_0\left(i\right) = \mathrm{id}_I\left(i\right),$$

and

$$\begin{aligned}
&\nu_{\bar{b}_1}\left(\nu_{\bar{b}_2}\left(i\right)\right) = \lambda_{s\left(\bar{b}_1\right)}\left(\lambda_{s\left(\bar{b}_2\right)}\left(i\right)\right) = \lambda_{s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)}\left(i\right) \\
&= \lambda_{s\left(\bar{b}_1 \circ \bar{b}_2\right)}\left(\lambda_{\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-}}\left(\lambda_{s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)}\left(i\right)\right)\right) = \lambda_{s\left(\bar{b}_1 \circ \bar{b}_2\right)}\left(\lambda_{\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)}\left(i\right)\right) \\
&= \lambda_{s\left(\bar{b}_1 \circ \bar{b}_2\right)}\left(\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) \circ \left(\left(s\left(\bar{b}_2\right)\right)^{-} \circ \left(s\left(\bar{b}_1\right)\right)^{-} \circ s\left(\bar{b}_1 \circ \bar{b}_2\right)\right.\right. \\
&\left.\left. + i\right)\right) = \lambda_{s\left(\bar{b}_1 \circ \bar{b}_2\right)}\left(i\right) = \nu_{\bar{b}_1 \circ \bar{b}_2}\left(i\right),
\end{aligned}$$

since $\left(s\left(\bar{b}_2\right)\right)^{-} \circ \left(s\left(\bar{b}_1\right)\right)^{-} \circ s\left(\bar{b}_1 \circ \bar{b}_2\right) \in I$ and so

$$\begin{aligned}
&\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) \circ \left(\left(s\left(\bar{b}_2\right)\right)^{-} \circ \left(s\left(\bar{b}_1\right)\right)^{-} \circ s\left(\bar{b}_1 \circ \bar{b}_2\right) + i\right) \\
&= \left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) \circ \left(\left(s\left(\bar{b}_2\right)\right)^{-} \circ \left(s\left(\bar{b}_1\right)\right)^{-} \circ s\left(\bar{b}_1 \circ \bar{b}_2\right) \circ i\right) = i.
\end{aligned}$$

Further if $\bar{b} \in \bar{B}$ and $i_1, i_2 \in I$, then

$$\nu_{\bar{b}}\left(i_1 + i_2\right) = \lambda_{s\left(\bar{b}\right)}\left(i_1 + i_2\right) = \lambda_{s\left(\bar{b}\right)}\left(i_1\right) + \lambda_{s\left(\bar{b}\right)}\left(i_2\right) = \nu_{\bar{b}}\left(i_1\right) + \nu_{\bar{b}}\left(i_2\right),$$

by Proposition 1.2.1. Hence ν is a left action from $\left(\bar{B}, \circ\right)$ into the group of automorphisms of $(I, +)$.
If $\bar{b} \in \bar{B}$, then

$$\begin{aligned}
\tau\left(\bar{b}, \bar{0}\right) &= s\left(\bar{b}\right) + s\left(\bar{0}\right) - s\left(\bar{b} + \bar{0}\right) = s\left(\bar{b}\right) - s\left(\bar{b}\right) = 0, \\
\tau\left(\bar{0}, \bar{b}\right) &= s\left(\bar{0}\right) + s\left(\bar{b}\right) - s\left(\bar{0} + \bar{b}\right) = s\left(\bar{b}\right) - s\left(\bar{b}\right) = 0,
\end{aligned}$$

further if $\bar{b}_1, \bar{b}_2, \bar{b}_3 \in \bar{B}$, then

$$\begin{aligned}
&\tau\left(\bar{b}_1+\bar{b}_2, \bar{b}_3\right)+\tau\left(\bar{b}_1, \bar{b}_2\right) \\
&=s\left(\bar{b}_1+\bar{b}_2\right)+s\left(\bar{b}_3\right)-s\left(\bar{b}_1+\bar{b}_2+\bar{b}_3\right)+s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)-s\left(\bar{b}_1+\bar{b}_2\right) \\
&=s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)-s\left(\bar{b}_1+\bar{b}_2\right)+s\left(\bar{b}_1+\bar{b}_2\right)+s\left(\bar{b}_3\right)-s\left(\bar{b}_1+\bar{b}_2+\bar{b}_3\right) \\
&=s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)+s\left(\bar{b}_3\right)-s\left(\bar{b}_1+\bar{b}_2+\bar{b}_3\right) \\
&=s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)+s\left(\bar{b}_3\right)-s\left(\bar{b}_2+\bar{b}_3\right)+s\left(\bar{b}_2+\bar{b}_3\right)-s\left(\bar{b}_1+\bar{b}_2+\bar{b}_3\right) \\
&=s\left(\bar{b}_2\right)+s\left(\bar{b}_3\right)-s\left(\bar{b}_2+\bar{b}_3\right)+s\left(\bar{b}_1\right)+s\left(\bar{b}_2+\bar{b}_3\right)-s\left(\bar{b}_1+\bar{b}_2+\bar{b}_3\right) \\
&=\tau\left(\bar{b}_2, \bar{b}_3\right)+\tau\left(\bar{b}_1, \bar{b}_2+\bar{b}_3\right).
\end{aligned}$$

Hence τ is a 2-cocycle from $\left(\bar{B},+\right)$ with values in $(I,+)$.
Moreover if $\bar{b} \in \bar{B}$, then

$$\begin{aligned}
\theta\left(\bar{b}, \bar{0}\right)&=\left(s\left(\bar{b} \circ \bar{0}\right)\right)^{-} \circ s\left(\bar{b}\right) \circ s\left(\bar{0}\right)=\left(s\left(\bar{b}\right)\right)^{-} \circ s\left(\bar{b}\right)=0 \\
\theta\left(\bar{0}, \bar{b}\right)&=\left(s\left(\bar{0} \circ \bar{b}\right)\right)^{-} \circ s\left(\bar{0}\right) \circ s\left(\bar{b}\right)=\left(s\left(\bar{b}\right)\right)^{-} \circ s\left(\bar{b}\right)=0
\end{aligned}$$

and if $\bar{b}_1, \bar{b}_2, \bar{b}_3 \in \bar{B}$, then

$$\begin{aligned}
&\sigma_{\bar{b}_3}\left(\theta\left(\bar{b}_1, \bar{b}_2\right)\right)+\theta\left(\bar{b}_1 \circ \bar{b}_2, \bar{b}_3\right) \\
&=\left(s\left(\bar{b}_3\right)\right)^{-} \circ\left(\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)\right) \circ s\left(\bar{b}_3\right) \\
&+\left(s\left(\bar{b}_1 \circ \bar{b}_2 \circ \bar{b}_3\right)\right)^{-} \circ s\left(\bar{b}_1 \circ \bar{b}_2\right) \circ s\left(\bar{b}_3\right) \\
&=\left(s\left(\bar{b}_1 \circ \bar{b}_2 \circ \bar{b}_3\right)\right)^{-} \circ s\left(\bar{b}_1 \circ \bar{b}_2\right) \circ s\left(\bar{b}_3\right) \\
&+\left(s\left(\bar{b}_3\right)\right)^{-} \circ\left(\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)\right) \circ s\left(\bar{b}_3\right) \\
&=\left(s\left(\bar{b}_1 \circ \bar{b}_2 \circ \bar{b}_3\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) \circ s\left(\bar{b}_3\right) \\
&=\left(s\left(\bar{b}_1 \circ \bar{b}_2 \circ \bar{b}_3\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2 \circ \bar{b}_3\right) \circ\left(s\left(\bar{b}_2 \circ \bar{b}_3\right)\right)^{-} \circ s\left(\bar{b}_2\right) \circ s\left(\bar{b}_3\right) \\
&=\left(s\left(\bar{b}_1 \circ \bar{b}_2 \circ \bar{b}_3\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2 \circ \bar{b}_3\right)+\left(s\left(\bar{b}_2 \circ \bar{b}_3\right)\right)^{-} \circ s\left(\bar{b}_2\right) \circ s\left(\bar{b}_3\right) \\
&=\theta\left(\bar{b}_1, \bar{b}_2 \circ \bar{b}_3\right)+\theta\left(\bar{b}_2, \bar{b}_3\right).
\end{aligned}$$

Further ν and σ satisfy the condition (1.10). In fact, since

$$\begin{aligned}
\nu_{\bar{b}}\left(\sigma_{\bar{b}}(i)\right)=\lambda_{s(\bar{b})}\left(\left(s\left(\bar{b}\right)\right)^{-} \circ i \circ s\left(\bar{b}\right)\right)&=-s\left(\bar{b}\right)+s\left(\bar{b}\right) \circ\left(s\left(\bar{b}\right)\right)^{-} \circ i \circ s\left(\bar{b}\right) \\
&=-s\left(\bar{b}\right)+i \circ s\left(\bar{b}\right),
\end{aligned}$$

and also, since I is the centre of $(B,+)$,

$$\begin{aligned}
&\nu_{\bar{b}}\left(\sigma_{\bar{b}}\left(i\right)\right)=\lambda_{s(\bar{b})}\left(\left(s\left(\bar{b}\right)\right)^{-}\circ i\circ s\left(\bar{b}\right)\right)\\
&=s\left(\bar{b}\right)\circ\left(\left(s\left(\bar{b}\right)\right)^{-}+\left(s\left(\bar{b}\right)\right)^{-}\circ i\circ s\left(\bar{b}\right)\right)\\
&=s\left(\bar{b}\right)\circ\left(\left(s\left(\bar{b}\right)\right)^{-}\circ i\circ s\left(\bar{b}\right)+\left(s\left(\bar{b}\right)\right)^{-}\right)=i\circ s\left(\bar{b}\right)-s\left(\bar{b}\right),
\end{aligned}$$

for every $\bar{b}\in\bar{B}$ and $i\in I$. Hence if $\bar{b}_1,\bar{b}_2\in\bar{B}$ and $i\in I$, then

$$\begin{aligned}
&\nu_{\bar{b}_1}\left(\sigma_{\bar{b}_1}\left(i\right)\right)-i+\nu_{\bar{b}_2}\left(\sigma_{\bar{b}_2}\left(i\right)\right)=-s\left(\bar{b}_1\right)+i\circ s\left(\bar{b}_1\right)-i+i\circ s\left(\bar{b}_2\right)-s\left(\bar{b}_2\right)\\
&=-s\left(\bar{b}_1\right)+i\circ\left(s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)\right)-s\left(\bar{b}_2\right)\\
&=-s\left(\bar{b}_1\right)+i\circ\left(s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)-s\left(\bar{b}_1+\bar{b}_2\right)+s\left(\bar{b}_1+\bar{b}_2\right)\right)-s\left(\bar{b}_2\right)\\
&=-s\left(\bar{b}_1\right)+i\circ\left(s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)-s\left(\bar{b}_1+\bar{b}_2\right)\right)-i+i\circ s\left(\bar{b}_1+\bar{b}_2\right)-s\left(\bar{b}_2\right)\\
&=-s\left(\bar{b}_1\right)+i+s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)-s\left(\bar{b}_1+\bar{b}_2\right)-i+i\circ s\left(\bar{b}_1+\bar{b}_2\right)-s\left(\bar{b}_2\right)\\
&=s\left(\bar{b}_2\right)-s\left(\bar{b}_1+\bar{b}_2\right)+i\circ s\left(\bar{b}_1+\bar{b}_2\right)-s\left(\bar{b}_2\right)\\
&=s\left(\bar{b}_2\right)+\nu_{\bar{b}_1+\bar{b}_2}\left(\sigma_{\bar{b}_1+\bar{b}_2}\left(i\right)\right)-s\left(\bar{b}_2\right)\\
&=\nu_{\bar{b}_1+\bar{b}_2}\left(\sigma_{\bar{b}_1+\bar{b}_2}\left(i\right)\right)+s\left(\bar{b}_2\right)-s\left(\bar{b}_2\right)=\nu_{\bar{b}_1+\bar{b}_2}\left(\sigma_{\bar{b}_1+\bar{b}_2}\left(i\right)\right),
\end{aligned}$$

i.e., $\nu_{\bar{b}_1}\left(\sigma_{\bar{b}_1}\left(i\right)\right)+\nu_{\bar{b}_2}\left(\sigma_{\bar{b}_2}\left(i\right)\right)=\nu_{\bar{b}_1+\bar{b}_2}\left(\sigma_{\bar{b}_1+\bar{b}_2}\left(i\right)\right)+i$.
Finally, ν,θ,τ satisfy condition (1.11). Note that $\nu_{\bar{b}_1\circ\bar{b}_2}\left(\theta\left(\bar{b}_1,\bar{b}_2\right)\right)=-s\left(\bar{b}_1\circ\bar{b}_2\right)+s\left(\bar{b}_1\right)\circ s\left(\bar{b}_2\right)$ and

$$\begin{aligned}
&\nu_{\bar{b}_1\circ\bar{b}_2}\left(\theta\left(\bar{b}_1,\bar{b}_2\right)\right)=\lambda_{s\left(\bar{b}_1\circ\bar{b}_2\right)}\left(\left(s\left(\bar{b}_1\circ\bar{b}_2\right)\right)^{-}\circ s\left(\bar{b}_1\right)\circ s\left(\bar{b}_2\right)\right)\\
&=s\left(\bar{b}_1\circ\bar{b}_2\right)\circ\left(\left(s\left(\bar{b}_1\circ\bar{b}_2\right)\right)^{-}+\left(s\left(\bar{b}_1\circ\bar{b}_2\right)\right)^{-}\circ s\left(\bar{b}_1\right)\circ s\left(\bar{b}_2\right)\right)\\
&=s\left(\bar{b}_1\circ\bar{b}_2\right)\circ\left(\left(s\left(\bar{b}_1\circ\bar{b}_2\right)\right)^{-}\circ s\left(\bar{b}_1\right)\circ s\left(\bar{b}_2\right)+\left(s\left(\bar{b}_1\circ\bar{b}_2\right)\right)^{-}\right)\\
&=s\left(\bar{b}_1\right)\circ s\left(\bar{b}_2\right)-s\left(\bar{b}_1\circ\bar{b}_2\right),
\end{aligned}$$

for all $\bar{b}_1, \bar{b}_2 \in \bar{B}$, since I is in the centre of $(B, +)$. Then, if $\bar{b}_1, \bar{b}_2, \bar{b}_3 \in \bar{B}$

$$\begin{aligned}
&\nu_{\bar{b}_1 \circ \bar{b}_2}\left(\theta\left(\bar{b}_1, \bar{b}_2\right)\right) + \nu_{\bar{b}_1 \circ \bar{b}_3}\left(\theta\left(\bar{b}_1, \bar{b}_3\right)\right) - \tau\left(\bar{b}_1, -\bar{b}_1 + \bar{b}_1 \circ \bar{b}_3\right) \\
&+ \tau\left(\bar{b}_1 \circ \bar{b}_2, -\bar{b}_1 + \bar{b}_1 \circ \bar{b}_3\right) = -s\left(\bar{b}_1 \circ \bar{b}_2\right) + s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) + s\left(\bar{b}_1\right) \circ s\left(\bar{b}_3\right) \\
&- s\left(\bar{b}_1 \circ \bar{b}_3\right) + s\left(\bar{b}_1 \circ \bar{b}_3\right) - s\left(-\bar{b}_1 + \bar{b}_1 \circ \bar{b}_3\right) - s\left(\bar{b}_1\right) + s\left(\bar{b}_1 \circ \bar{b}_2\right) \\
&+ s\left(-\bar{b}_1 + \bar{b}_1 \circ \bar{b}_3\right) - s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) = -s\left(\bar{b}_1 \circ \bar{b}_2\right) + s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) \\
&+ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_3\right) - s\left(-\bar{b}_1 + \bar{b}_1 \circ \bar{b}_3\right) - s\left(\bar{b}_1\right) + s\left(\bar{b}_1\right) \circ s\left(\bar{b}_3\right) \\
&- s\left(\bar{b}_1\right) \circ s\left(\bar{b}_3\right) + s\left(\bar{b}_1 \circ \bar{b}_2\right) + s\left(-\bar{b}_1 + \bar{b}_1 \circ \bar{b}_3\right) - s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) \\
&= s\left(\bar{b}_1\right) \circ s\left(\bar{b}_3\right) - s\left(\bar{b}_1\right) \circ s\left(\bar{b}_3\right) + s\left(\bar{b}_1 \circ \bar{b}_2\right) - s\left(\bar{b}_1 \circ \bar{b}_2\right) + s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) \\
&+ s\left(-\bar{b}_1 + \bar{b}_1 \circ \bar{b}_3\right) - s\left(-\bar{b}_1 + \bar{b}_1 \circ \bar{b}_3\right) - s\left(\bar{b}_1\right) + s\left(\bar{b}_1\right) \circ s\left(\bar{b}_3\right) \\
&- s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) = s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) - s\left(\bar{b}_1\right) + s\left(\bar{b}_1\right) \circ s\left(\bar{b}_3\right) \\
&- s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) = s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right)\right) - s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) \\
&= s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right)\right) - s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right)\right) \\
&- s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right)\right) = -s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right)\right) \\
&+ s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right)\right) - s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right)\right) = -s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) \\
&+ s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right)\right) = \nu_{\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)}\left(\theta\left(\bar{b}_1, \bar{b}_2 + \bar{b}_3\right)\right) + \nu_{\bar{b}_1}\left(\tau\left(\bar{b}_2, \bar{b}_3\right)\right),
\end{aligned}$$

since

$$\begin{aligned}
&\nu_{\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)}\left(\theta\left(\bar{b}_1, \bar{b}_2 + \bar{b}_3\right)\right) + \nu_{\bar{b}_1}\left(\tau\left(\bar{b}_2, \bar{b}_3\right)\right) \\
&= \lambda_{s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right)}\left(\left(s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2 + \bar{b}_3\right)\right) \\
&+ \lambda_{s\left(\bar{b}_1\right)}\left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right) - s\left(\bar{b}_2 + \bar{b}_3\right)\right) = -s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) \\
&+ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2 + \bar{b}_3\right) - s\left(\bar{b}_1\right) + s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right) - s\left(\bar{b}_2 + \bar{b}_3\right)\right) \\
&= -s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2 + \bar{b}_3\right) + s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right) - s\left(\bar{b}_2 + \bar{b}_3\right)\right) \\
&= -s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right) - s\left(\bar{b}_2 + \bar{b}_3\right) + s\left(\bar{b}_2 + \bar{b}_3\right)\right) \\
&= -s\left(\bar{b}_1 \circ \left(\bar{b}_2 + \bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ \left(s\left(\bar{b}_2\right) + s\left(\bar{b}_3\right)\right).
\end{aligned}$$

Hence we may consider the extension of the skew left brace $\bar{B}$ with the zero left brace I. Consider the map $\psi : \bar{B} \times I \to B$ defined by $\psi\left(\bar{b}, i\right) := s\left(\bar{b}\right) + i$, for every

$b \in B$, where $b = s\left(\bar{b}\right) + i$. We may check that ψ is an isomorphism. In fact ψ is bijective and if $\bar{b}_1, \bar{b}_2 \in \bar{B}$ and $i_1, i_2 \in I$, then

$$\begin{aligned}
&\psi\left(\bar{b}_1, i_1\right) + \psi\left(\bar{b}_2, i_2\right) = s\left(\bar{b}_1\right) + i_1 + s\left(\bar{b}_2\right) + i_2 \\
&= s\left(\bar{b}_1\right) + s\left(\bar{b}_2\right) - s\left(\bar{b}_1 + \bar{b}_2\right) + s\left(\bar{b}_1 + \bar{b}_2\right) + i_1 + i_2 \\
&= s\left(\bar{b}_1 + \bar{b}_2\right) + i_1 + i_2 + \tau\left(\bar{b}_1, \bar{b}_2\right) = \psi\left(\bar{b}_1 + \bar{b}_2, i_1 + i_2 + \tau\left(\bar{b}_1, \bar{b}_2\right)\right) \\
&= \psi\left(\left(\bar{b}_1, i_1\right) + \left(\bar{b}_2, i_2\right)\right)
\end{aligned}$$

since $\tau\left(\bar{b}_1, \bar{b}_2\right) \in I$ and I is in the centre of $(B, +)$. Moreover note that, if $\bar{b}_1, \bar{b}_2 \in \bar{B}$ and $i \in I$, then

$$\begin{aligned}
&\lambda_{s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)}(i) = \lambda_{s\left(\bar{b}_1 \circ \bar{b}_2\right) \circ \left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)}(i) \\
&= \lambda_{s\left(\bar{b}_1 \circ \bar{b}_2\right)}\left(\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) \circ \left(\left(\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)\right)^{-}\right.\right. \\
&\left.\left. + i\right)\right) = \lambda_{s\left(\bar{b}_1 \circ \bar{b}_2\right)}(i),
\end{aligned}$$

since $\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right), i \in I$ and I is a zero left brace. Hence we have

$$\begin{aligned}
&\psi\left(\left(\bar{b}_1, i_1\right) \circ \left(\bar{b}_2, i_2\right)\right) = \psi\left(\left(\bar{b}_1 \circ \bar{b}_2, \nu_{\bar{b}_1 \circ \bar{b}_2}\left(\sigma_{\bar{b}_2}\left(\nu_{\bar{b}_1^{-}}(i_1)\right)\right) + \nu_{\bar{b}_1}(i_2) + \nu_{\bar{b}_1 \circ \bar{b}_2}\left(\theta\left(\bar{b}_1, \bar{b}_2\right)\right)\right)\right) \\
&= s\left(\bar{b}_1 \circ \bar{b}_2\right) + \lambda_{s\left(\bar{b}_1 \circ \bar{b}_2\right)}\left(\left(s\left(\bar{b}_2\right)\right)^{-} \circ \lambda^{-1}_{s\left(\bar{b}_1\right)}(i_1) \circ s\left(\bar{b}_2\right)\right) + \lambda_{s\left(\bar{b}_1\right)}(i_2) \\
&+ \lambda_{s\left(\bar{b}_1 \circ \bar{b}_2\right)}\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)\right) \\
&= s\left(\bar{b}_1 \circ \bar{b}_2\right) \circ \left(\left(s\left(\bar{b}_2\right)\right)^{-} \circ \lambda^{-1}_{s\left(\bar{b}_1\right)}(i_1) \circ s\left(\bar{b}_2\right)\right. \\
&\left. + \lambda_{\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ \left(\bar{b}_1\right)}(i_2) + s\left(\bar{b}_1 \circ \bar{b}_2\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)\right) \\
&= s\left(\bar{b}_1 \circ \bar{b}_2\right) \circ \left(s\left(\bar{b}_1 \circ \bar{b}_2\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)\right. \\
&\left. + \left(s\left(\bar{b}_2\right)\right)^{-} \circ \lambda^{-1}_{s\left(\bar{b}_1\right)}(i_1) \circ s\left(\bar{b}_2\right) + \lambda_{\left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ \left(\bar{b}_1\right)}(i_2)\right) \\
&= s\left(\bar{b}_1 \circ \bar{b}_2\right) \circ s\left(\bar{b}_1 \circ \bar{b}_2\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) \\
&\circ \left(s\left(\bar{b}_2\right)\right)^{-} \circ \lambda^{-1}_{s\left(\bar{b}_1\right)}(i_1) \circ s\left(\bar{b}_2\right) \circ \lambda_{\left(s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)\right)^{-} \circ \left(\bar{b}_1\right)}(i_2) \\
&= s\left(\bar{b}_1\right) \circ \lambda_{s\left(\bar{b}_1\right)}(i_1) \circ s\left(\bar{b}_2\right) \circ \lambda_{\left(\bar{b}_2\right)^{-}}(i_2) \\
&= \left(s\left(\bar{b}_1\right) + i_1\right) \circ \left(s\left(\bar{b}_2\right) + i_2\right) = \psi\left(s\left(\bar{b}_1\right), i_1\right) \circ \psi\left(s\left(\bar{b}_2\right), i_2\right)
\end{aligned}$$

□

Remark 1.3.22. In the previous construction, $\{0\} \times I$ lies in the socle of $B \times I$ if and only if $\nu_b(\sigma_b(i)) = i$ for all $i \in I$ and $b \in B$. In fact $(0,i) \in \{0\} \times I$ lies in $\operatorname{Soc}(B \times I)$ if and only if $(0,i) + (b,j) = (0,i) \circ (b,j)$, i.e.,

$$(b, i+j) = (b, \nu_b(\sigma_b(i)) + j),$$

for every $(b,j) \in B \times I$. In this case the condition (1.10) becomes $\nu_b(\sigma_b(i)) = i$, for all $b \in B$ and $i \in I$ and the product is given by

$$\begin{aligned}(b_1,i_1) \circ (b_2,i_2) &= \left(b_1 \circ b_2, \nu_{b_1}\left(\nu_{b_2}\left(\sigma_{b_2}\left(\nu_{b_1^-}(i_1)\right)\right)\right) + \nu_{b_1}(i_2) + \nu_{b_1 \circ b_2}(\theta(b_1,b_2))\right)\\ &= \left(b_1 \circ b_2, \nu_{b_1}\left(\nu_{b_1^-}(i_1)\right) + \nu_{b_1}(i_2) + \nu_{b_1 \circ b_2}(\theta(b_1,b_2))\right)\\ &= (b_1 \circ b_2, i_1 + \nu_{b_1}(i_2) + \nu_{b_1 \circ b_2}(\theta(b_1,b_2))),\end{aligned}$$

for all $(b_1,i_1), (b_2,i_2) \in B \times I$.

Recall that if $(B,+)$, $(I,+)$ are abelian groups, then a map $\tau : B \times B \to I$ is a *symmetric 2-cocycle from* $(B,+)$ *with values in* $(I,+)$ if τ is a 2-cocycle and moreover the following condition holds:

3. $\tau(b_1,b_2) = \tau(b_2,b_1)$, for all $b_1, b_2 \in B$.

Corollary 1.3.23 (Theorem 3.3 in [2])**.** *Let B be a left brace, I a zero left brace, $\sigma : B \to \operatorname{Aut}(I)$ a right action from $(B,\circ)$ into the group of automorphisms of $(I,+)$, $\nu : B \to \operatorname{Aut}(I)$ a left action from $(B,\circ)$ into the group of automorphisms of $(I,+)$, $\tau : B \times B \to I$ a* symmetric 2*-cocycle from $(B,+)$ with values in $(I,+)$ and $\theta : B \times B \to I$ a* 2*-cocycle from $(B,\circ)$ with values in $(I,+)$ respect to σ. If the following conditions hold:*

$$\nu_{b_1+b_2}(\sigma_{b_1+b_2}(i)) + i = \nu_{b_1}(\sigma_{b_1}(i)) + \nu_{b_2}(\sigma_{b_2}(i)); \tag{1.12}$$

$$\begin{aligned}&\nu_{b_1 \circ (b_2+b_3)}(\theta(b_1, b_2+b_3)) + \nu_{b_1}(\tau(b_2,b_3)) + \tau(b_1 \circ (b_2+b_3), b_1)\\ &= \nu_{b_1 \circ b_2}(\theta(b_1,b_2)) + \nu_{b_1 \circ b_3}(\theta(b_1,b_3)) + \tau(b_1 \circ b_2,\ b_1 \circ b_3),\end{aligned} \tag{1.13}$$

for all $b, b_1, b_2, b_3 \in B$ and $i \in I$, then $B \times I$ is a left brace with the sum

$$(b_1,i_1) + (b_2,i_2) := (b_1+b_2, i_1+i_2+\tau(b_1,b_2))$$

and the product

$$(b_1,i_1) \circ (b_2,i_2) := \left(b_1 \circ b_2,\ \nu_{b_1 \circ b_2}\left(\sigma_{b_2}\left(\nu_{b_1^-}(i_1)\right)\right) + \nu_{b_1}(i_2) + \nu_{b_1 \circ b_2}(\theta(b_1,b_2))\right),$$

for all $(b_1,i_1), (b_2,i_2) \in B \times I$. Moreover, $\{0\} \times I$ is an ideal of $B \times I$ isomorphic to I such that $(B \times I)/(\{0\} \times I) \cong B$.

Conversely, if B is a left brace, I is an ideal of B such that I is a trivial left brace and $\bar{B} := B/I$, then B is isomorphic to a left brace of the form described above, constructed from $\bar{B}$ and I.

Proof. First note that (1.13) is equivalent to (1.11). In fact if $b_1, b_2, b_3 \in B$, then

$$\begin{aligned}
&-\tau(b_1, -b_1 + b_1 \circ b_3) + \tau(b_1 \circ b_2, -b_1 + b_1 \circ b_3) \\
&= -\tau(b_1, -b_1 + b_1 \circ b_3) + \tau(b_1 \circ b_2, -b_1 + b_1 \circ b_3 + b_1) \\
&\quad + \tau(-b_1 + b_1 \circ b_3, b_1) - \tau(b_1 \circ b_2 - b_1 + b_1 \circ b_3, b_1) \\
&= -\tau(b_1, -b_1 + b_1 \circ b_3) + \tau(b_1 \circ b_2, b_1 \circ b_3) \\
&\quad + \tau(b_1, -b_1 + b_1 \circ b_3) - \tau(b_1 \circ (b_2 + b_3), b_1) \\
&= \tau(b_1 \circ b_2, b_1 \circ b_3) - \tau(b_1 \circ (b_2 + b_3), b_1).
\end{aligned}$$

Hence

$$\begin{aligned}
&\nu_{b_1 \circ (b_2 + b_3)}(\theta(b_1, b_2 + b_3)) + \nu_{b_1}(\tau(b_2, b_3)) = \nu_{b_1 \circ b_2}(\theta(b_1, b_2)) + \nu_{b_1 \circ b_3}(\theta(b_1, b_3)) \\
&+ \tau(b_1 \circ b_2,\ b_1 \circ b_3) - \tau(b_1 \circ (b_2 + b_3), b_1) = \nu_{b_1 \circ b_2}(\theta(b_1, b_2)) + \nu_{b_1 \circ b_3}(\theta(b_1, b_3)) \\
&- \tau(b_1, -b_1 + b_1 \circ b_3) + \tau(b_1 \circ b_2, -b_1 + b_1 \circ b_3),
\end{aligned}$$

for all $b_1, b_2, b_3 \in B$. Therefore, by Theorem 1.3.21, $(B \times I, +, \circ)$ is a skew left brace and $\{0\} \times I$ is an ideal of $B \times I$ isomorphic to I such that $(B \times I) / (\{0\} \times I) \cong I$. Further

$$\begin{aligned}
(b_1, i_1) + (b_2, i_2) &= (b_1 + b_2, i_1 + i_2 + \tau(b_1, b_2)) = (b_2 + b_1, i_2 + i_1 + \tau(b_2, b_1)) \\
&= (b_2, i_2) + (b_2, i_2),
\end{aligned}$$

for all $b_1, b_2 \in B$ and $i_1, i_2 \in I$, i.e., $(B \times I, +)$ is an abelian group ad so $(B \times I, +, \circ)$ is a left brace.
On the other hand let B be a left brace, I an ideal of B with zero structure. As in Theorem 1.3.21, set $\bar{B} := B/I$ and consider $\pi : B \to \bar{B}$ the projection map, $s : \bar{B} \to B$ a map such that $s(\bar{0}) = 0$ and $\pi\left(s\left(\bar{b}\right)\right) = \bar{b}$, for every $\bar{b} \in \bar{B}$. Consider maps $\tau : \bar{B} \times \bar{B} \to I$ defined by $\tau\left(\bar{b}_1, \bar{b}_2\right) := s\left(\bar{b}_1\right) + s\left(\bar{b}_2\right) - s\left(\bar{b}_1 + \bar{b}_2\right)$, for all $\bar{b}_1, \bar{b}_2 \in \bar{B}$, $\theta : \bar{B} \times \bar{B} \to I$ defined by $\theta\left(\bar{b}_1, \bar{b}_2\right) := \left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^- \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)$, for all $\bar{b}_1, \bar{b}_2 \in \bar{B}$, $\sigma : \bar{B} \to \mathrm{Aut}(I)$ defined by $\sigma_{\bar{b}} : I \to I, i \mapsto \left(s\left(\bar{b}\right)\right)^- \circ i \circ s\left(\bar{b}\right)$, for every $\bar{b} \in \bar{B}$ and $\nu : \bar{B} \to \mathrm{Aut}(I)$ defined by $\nu_{\bar{b}} : I \to I, i \mapsto \lambda_{s\left(\bar{b}\right)}(i)$, for every $\bar{b} \in \bar{B}$. Then $\bar{B}, I, \tau, \theta, \sigma, \nu$ satisfy Theorem 1.3.21. Moreover $\bar{B}$ is a brace and

$$\begin{aligned}
\tau\left(\bar{b}_1, \bar{b}_2\right) &= s\left(\bar{b}_1\right) + s\left(\bar{b}_2\right) - s\left(\bar{b}_1 + \bar{b}_2\right) = s\left(\bar{b}_2\right) + s\left(\bar{b}_1\right) - s\left(\bar{b}_2 + \bar{b}_1\right) \\
&= \tau\left(\bar{b}_2, \bar{b}_1\right),
\end{aligned}$$

for all $b_1, b_2 \in B$, i.e., τ is a symmetric 2-cocycle. So $\bar{B}, I, \tau, \theta, \sigma, \nu$ satisfy condition (1.13). Therefore we may consider the left brace obtained from $\bar{B}$ and I and we have that it is isomorphic to B. □

1.3.2 Skew left braces with non-trivial annihilator

By Theorem 1.3.21, we may describe all skew left braces with non trivial annihilator. Note that if B is a skew left brace then the annihilator is the ideal

$$\operatorname{Ann}(B) = \{a \mid a \in B \quad \forall b \in B \quad a \circ b = a + b = b + a = b \circ a\}.$$

Definition 1.3.24. Let $(B, +, \circ)$ be a skew left brace, $(I, +)$ an abelian group. A pair (τ, θ) such that $\tau : B \times B \to I$ is a 2-cocycle from $(B, +)$ with values in I, $\theta : B \times B \to A$ is a 2-cocycle of $(B, \circ)$ with values in I and they satisfy

$$\begin{aligned} \theta(b_1, b_2 + b_3) + \tau(b_2, b_3) = \quad & \theta(b_1, b_2) + \theta(b_1, b_3) - \tau(b_1, -b_1 + b_1 \circ b_3) \\ & + \tau(b_1 \circ b_2, -b_1 + b_1 \circ b_3), \end{aligned} \tag{1.14}$$

for all $b_1, b_2, b_3 \in B$, is said a *Hochschild pair of the skew left brace B with values in I*.

Proposition 1.3.25. *Let $(B, +, \circ)$ be a skew left brace, $(I, +)$ an abelian group and (τ, θ) a Hochschild pair of B with values in I. If we define on the cartesian product $B \times I$*

$$(b_1, i_1) + (b_2, i_2) := (b_1 + b_2, i_1 + i_2 + \tau(b_1, b_2)), \tag{1.15}$$

$$(b_1, i_1) \circ (b_2, i_2) := (b_1 \circ b_2,\ i_1 + i_2 + \theta(b_1, b_2)), \tag{1.16}$$

for all $b_1, b_2 \in B$, $i_1, i_2 \in I$, then $(B \times I, +, \circ)$ is a skew left brace.

Proof. Since $(I, +)$ is an abelian group we may review it as a zero left brace, moreover consider $\sigma : B \to \operatorname{Aut}(I)$ and $\nu : B \to \operatorname{Aut}(I)$ as trivial actions. Then $\theta, \sigma, \nu, \tau$ satisfy the conditions of Theorem 1.3.21. In fact θ is a 2-cocycle of $(B, \circ)$ with values in $(I, +)$ respect to the trivial action σ, further τ is a 2-cocycle of $(B, +)$ with values in $(I, +)$ by assumption. Clearly, conditions (1.10) and (1.11) hold. Therefore we may consider the extension of the skew left brace B by I where the sum and the product are given by

$$\begin{aligned} (b_1, i_1) + (b_2, i_2) &= (b_1 + b_2,\ i_1 + i_2 + \tau(b_1, b_2)), \\ (b_1, i_1) \circ (b_2, i_2) &= (b_1 \circ b_2,\ i_1 + i_2 + \theta(b_1, b_2)), \end{aligned}$$

i.e., the thesis. □

Definition 1.3.26. Let $(B,+,\circ)$ be a skew left brace, $(I,+)$ an abelian group, (τ,θ) a Hochschild pair of the skew left brace B with values in I. We call the skew left brace constructed in previous proposition the *Hochschild product of B by I (via τ and θ)*.

Theorem 1.3.27. *Let B be a skew left brace such that $\mathrm{Ann}\,(B)\neq 0$ and $I:=\mathrm{Ann}\,(B)$. Then there exists a Hochschild pair (τ,θ) of the skew left brace $\bar{B}:=B/I$ with values in I such that B is isomorphic to the Hochschild product of $\bar{B}$ by I (via τ and θ).*

Proof. By proof of Theorem 1.3.21, consider $\pi: B\to\bar{B}$ the projection map and $s:\bar{B}\to B$ a map such that $s\left(\bar{0}\right)=0$ and $\pi\left(s\left(\bar{b}\right)\right)=\bar{b}$, for every $\bar{b}\in\bar{B}$. Then the right action $\sigma:\bar{B}\to\mathrm{Aut}\,(I)$ from $\left(\bar{B},\circ\right)$ into the group of automorphisms of $(I,+)$ is

$$\sigma_{\bar{b}}: I\to I,\quad i\mapsto\left(s\left(\bar{b}\right)\right)^{-}\circ i\circ s\left(\bar{b}\right)=i,$$

for every $\bar{b}\in\bar{B}$, since I is the annihilator of B and, in particular, I is in the centre of $(B,\circ)$. The left action $\nu:\bar{B}\to\mathrm{Aut}\,(I)$ from $\left(\bar{B},\circ\right)$ into the group of automorphisms of $(I,+)$ is

$$\nu_{\bar{b}}: I\to I,\quad i\mapsto\lambda_{s\left(\bar{b}\right)}\left(i\right)=-s\left(\bar{b}\right)+s\left(\bar{b}\right)\circ i=-s\left(\bar{b}\right)+s\left(\bar{b}\right)+i=i,$$

for every $\bar{b}\in\bar{B}$, since i lies in the annihilator of B. Moreover, the 2-cocycle from $\left(\bar{B},+\right)$ with values in $(I,+)$, $\tau:\bar{B}\times\bar{B}\to I$ is

$$\tau\left(\bar{b}_1,\bar{b}_2\right)=s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)-s\left(\bar{b}_1+\bar{b}_2\right),$$

for all $\bar{b}_1,\bar{b}_2\in\bar{B}$. Further, $\theta:\bar{B}\times\bar{B}\to A$, defined by

$$\theta\left(\bar{b}_1,\bar{b}_2\right)=\left(s\left(\bar{b}_1\circ\bar{b}_2\right)\right)^{-}\circ s\left(\bar{b}_1\right)\circ s\left(\bar{b}_2\right),$$

is a 2-cocycle from $\left(\bar{B},\circ\right)$ with values in I, since it is a 2-cocycle from $\left(\bar{B},\circ\right)$ with values in I respect to the trivial action. Hence (τ,θ) is a Hochschild pair and the extension of the skew left brace $\bar{B}$ by I is the Hochschild product of $\bar{B}$ by I (via τ and θ). Finally, $\psi:\bar{B}\times I\to B$, defined by $\psi\left(\bar{b},i\right)=s\left(\bar{b}\right)+i$, for every $\left(\bar{b},i\right)\in\bar{B}\times I$, is an isomorphism from $\bar{B}\times I$ into B . □

1.3.3 Left braces with non-trivial annihilator

We may describe all left braces with non-trivial annihilator.

Proposition 1.3.28. *Let $(B,+,\circ)$ be a left brace, $(I,+)$ an abelian group, (τ,θ) a Hochschild pair of the left brace B with values in I, with τ symmetric. Then the Hochschild product of B by I (via τ and θ) is a left brace.*

Proof. By Proposition 1.3.25, $(B\times I,+,\circ)$ is a skew left brace with the sum and the product given by

$$\begin{aligned}(b_1,i_1)+(b_2,i_2)&=(b_1+b_2,i_1+i_2+\tau(b_1,b_2)),\\(b_1,i_1)\circ(b_2,i_2)&=(b_1\circ b_2,i_1+i_2+\theta(b_1,b_2)),\end{aligned}$$

for all $b_1,b_2\in B$ and $i_1,i_2\in I$. Moreover $(B\times I,+)$ is an abelian group, since $(B,+)$ and $(I,+)$ are abelian and τ is a symmetric 2-cocycle. □

Theorem 1.3.29. *Let B be a left brace such that $\operatorname{Ann}(B)\neq 0$ and $I:=\operatorname{Ann}(B)$. Then there exists a Hochschild pair (τ,θ) of the left brace $\bar{B}:=B/I$ with values in I, with τ symmetric, such that B is isomorphic to the Hochschild product of $\bar{B}$ by I (via τ and θ).*

Proof. By Theorem 1.3.27, there exists a Hochschild pair (τ,θ) of the skew left brace $\bar{B}$ with values in I such that B is isomorphic to the Hochschild product of $\bar{B}$ by I.

Moreover τ is symmetric. In fact, if $\bar{b}_1,\bar{b}_2\in\bar{B}$ then

$$\begin{aligned}\tau\left(\bar{b}_1,\bar{b}_2\right)&=s\left(\bar{b}_1\right)+s\left(\bar{b}_2\right)-s\left(\bar{b}_1+\bar{b}_2\right)=s\left(\bar{b}_2\right)+s\left(\bar{b}_1\right)-s\left(\bar{b}_2+\bar{b}_1\right)\\&=\tau\left(\bar{b}_2,\bar{b}_1\right).\end{aligned}$$

□

1.3.4 Left F-braces with non-trivial annihilator

As consequence of previous results we may find the description of all left F-braces with non-trivial F-annihilator as in [9].

Definition 1.3.30 (see [9])**.** Let F be a field, B a left F-brace. The *F-annihilator* of B is the set

$$\operatorname{Ann}_F(B):=\{a\mid a\in B\ \forall b\in B\ \forall\mu\in F\quad(\mu a)\circ b=(\mu a)+b=b\circ(\mu a)\}.$$

Proposition 1.3.31. *Let F be a field and B a left F-brace. Then $\operatorname{Ann}_F(B)$ is an ideal of the left brace B, a subspace of the vector space B and it is contained in $\operatorname{Ann}(B)$.*

Proof. Clearly, $\operatorname{Ann}_F(B)$ is not empty, since $0 \in \operatorname{Ann}_F(B)$. If $i \in \operatorname{Ann}_F(B)$, then $i + b = i \circ b$, for every $b \in B$, i.e., $i \in \operatorname{Soc}(B)$, further $i \circ b = b \circ i$, i.e., $i \in \mathrm{Z}(B)$. Hence $\operatorname{Ann}_F(B) \subseteq \operatorname{Ann}(B)$. Moreover, if $i \in \operatorname{Ann}_F(B)$, then $-i = i^-$, so $(\mu i^-) \circ b = (-\mu i) \circ b = (-\mu i) + b = (\mu i^-) + b$, and $(\mu i^-) \circ b = (-\mu i) \circ b = b \circ (-\mu i) = b \circ (\mu i^-)$, for all $\mu \in F$ and $b \in B$. Hence $-i = i^- \in \operatorname{Ann}_F(B)$. If $i_1, i_2 \in \operatorname{Ann}_F(B)$, then

$$\begin{aligned}(\mu(i_1 + i_2)) + b &= (\mu i_1) + (\mu i_2) + b = (\mu i_1) + ((\mu i_2) \circ b) = (\mu i_1) \circ (\mu i_2) \circ b \\ &= ((\mu i_1) + (\mu i_2)) \circ b = (\mu(i_1 + i_2)) \circ b\end{aligned}$$

and

$$\begin{aligned}(\mu(i_1 + i_2)) \circ b &= (\mu(i_1 + i_2)) + b = (\mu i_1) + (\mu i_2) + b = ((\mu i_1) + b) \circ (\mu i_2) \\ &= b \circ (\mu i_1) \circ (\mu i_2) = b \circ ((\mu i_1) \circ (\mu i_2)) = b \circ ((\mu i_1) + (\mu i_2)) \\ &= b \circ (\mu(i_1 + i_2)),\end{aligned}$$

for all $\mu \in F$ and $b \in B$. Therefore $i_1 + i_2 \in \operatorname{Ann}_F(B)$ and $\operatorname{Ann}_F(B)$ is a subgroup of $(B, +)$. If $i_1, i_2 \in \operatorname{Ann}_F(B)$, then $i_1 \circ i_2 = i_1 + i_2$, and so $\operatorname{Ann}_F(B)$ is a subgroup of $(B, \circ)$. Further if $i \in \operatorname{Ann}_F(B)$, $b^- \circ i \circ b = b^- \circ b \circ i = i \in \operatorname{Ann}_F(B)$, for every $b \in B$, i.e., $\operatorname{Ann}_F(B)$ is a normal subgroup of $(B, \circ)$. Moreover, if $i \in \operatorname{Ann}_F(B)$, then

$$\rho_b(i) = \left(i^- + b\right)^- \circ b = \left(b \circ i^-\right)^- \circ b = i \circ b^- \circ b = i \in \operatorname{Ann}_F(B),$$

for every $b \in B$. Hence $\operatorname{Ann}_F(B)$ is an ideal of the brace B. Finally, if $i \in \operatorname{Ann}_F(B)$ and $\lambda \in F$, then

$$(\mu(\lambda i)) \circ b = ((\mu\lambda) i) \circ b = ((\mu\lambda) i) + b = (\mu(\lambda i)) + b$$

and

$$(\mu(\lambda i)) \circ b = ((\mu\lambda) i) + b = b \circ ((\mu\lambda) i) = b \circ (\mu(\lambda i)),$$

for all $\mu \in F$ and $b \in B$, i.e., $\operatorname{Ann}_F(B)$ is a subspace of the vector space B over the field F. □

Proposition 1.3.32. *Let F be a field, B a left F-brace and I an ideal and a subspace of B. Then B/I is a left F-brace. In particular, $\bar{B} := B/\operatorname{Ann}_F(B)$ is a left F-brace.*

Proof. We have that B/I is a left brace, since I is an ideal of B. Further B/I is a vector space over F, since I is a subspace of the vector space B. Finally, if $b_1, b_2 \in B$ and $\mu \in F$, then

$$\begin{aligned}&\mu\left(\left(b_1+I\right)\circ\left(b_2+I\right)\right)=\mu\left(\left(b_1\circ b_2\right)+I\right)=\left(\mu\left(b_1\circ b_2\right)\right)+I\\&=\left(b_1\circ\left(\mu b_2\right)+\left(\mu-1\right)b_1\right)+I=\left(b_1+I\right)\circ\left(\mu b_2+I\right)+\left(\left(\mu-1\right)b_1+I\right).\end{aligned}$$

Hence B/I is a left F-brace. □

Definition 1.3.33. Let F be a field, $(B,+,\circ)$ a left F-brace and $(V,+)$ a vector space over F. A map $\theta : B \times B \to V$ such that

1. $\theta\left(a,\mu b+\nu c\right)=\mu\theta\left(a,b\right)+\nu\theta\left(a,c\right)$,
2. $\theta\left(a\circ b,c\right)+\theta\left(a,b\right)=\theta\left(b,c\right)+\theta\left(a,b\circ c\right)$,

for all $a,b,c \in B$ and $\mu,\nu \in F$ is called 2-*cocycle of left F-brace B with values in V*.

In particular θ is a 2-cocycle of $(B,\circ)$ with values in V. In fact if $a \in B$, then

$$\theta\left(a,0\right)=\theta\left(a,0+0\right)=\theta\left(a,0\right)+\theta\left(a,0\right),$$

and so $\theta\left(a,0\right)=0$, since $(V,+)$ is a group, moreover

$$\theta\left(0,a\right)=\theta\left(0\circ 0,a\right)=-\theta\left(0,0\right)+\theta\left(0,0\circ a\right)+\theta\left(0,a\right)=\theta\left(0,a\right)+\theta\left(0,a\right),$$

i.e., $\theta\left(0,a\right)=0$.

Example 1.3.34. Let N be the zero algebra of dimension n over a field F, $(e_1,\ldots,e_n)$ a basis of N and τ an endomorphism of the additive group of F. The map $\theta : N \times N \to F$ such that

$$\theta\left(\sum_{i=1}^{n}x_ie_i,\sum_{i=1}^{n}y_ie_i\right)=\tau\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right),$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in F$ is a 2-cocycle of the left F-brace N, but it is a 2-cocyle of the F-algebra N if and only if τ is linear.
In fact, if $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n \in F$, then

$$\begin{aligned}
&\theta\left(\sum_{i=1}^{n} x_i e_i, \mu\sum_{i=1}^{n} y_i e_i + \nu\sum_{i=1}^{n} z_i e_i\right) = \theta\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} (\mu y_i + \nu z_i)\, e_i\right)\\
&= \tau\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} (\mu y_i + \nu z_i)\right) = \tau\left(\sum_{i=1}^{n} x_i\right)\left(\mu\sum_{i=1}^{n} y_i + \nu\sum_{i=1}^{n} z_i\right)\\
&= \mu\tau\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) + \nu\tau\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} z_i\right)\\
&= \mu\theta\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i\right) + \nu\theta\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} z_i e_i\right),
\end{aligned}$$

$$\begin{aligned}
&\theta\left(\left(\sum_{i=1}^{n} x_i e_i\right) \circ \left(\sum_{i=1}^{n} y_i e_i\right), \sum_{i=1}^{n} z_i e_i\right) + \theta\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i\right)\\
&= \theta\left(\left(\sum_{i=1}^{n} x_i e_i\right) + \left(\sum_{i=1}^{n} y_i e_i\right), \sum_{i=1}^{n} z_i e_i\right) + \theta\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i\right)\\
&= \theta\left(\sum_{i=1}^{n} (x_i + y_i)\, e_i, \sum_{i=1}^{n} z_i e_i\right) + \theta\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i\right)\\
&= \tau\left(\sum_{i=1}^{n} (x_i + y_i)\right)\left(\sum_{i=1}^{n} z_i\right) + \tau\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\\
&= \left(\tau\left(\sum_{i=1}^{n} x_i\right) + \tau\left(\sum_{i=1}^{n} y_i\right)\right)\left(\sum_{i=1}^{n} z_i\right) + \tau\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\\
&= \tau\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} z_i\right) + \tau\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} z_i\right) + \tau\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)
\end{aligned}$$

and

$$\begin{aligned}
&\theta\left(\sum_{i=1}^{n} y_i e_i, \sum_{i=1}^{n} z_i e_i\right) + \theta\left(\sum_{i=1}^{n} x_i e_i, \left(\sum_{i=1}^{n} y_i e_i\right) \circ \left(\sum_{i=1}^{n} z_i e_i\right)\right)\\
&= \theta\left(\sum_{i=1}^{n} y_i e_i, \sum_{i=1}^{n} z_i e_i\right) + \theta\left(\sum_{i=1}^{n} x_i e_i, \left(\sum_{i=1}^{n} y_i e_i\right) + \left(\sum_{i=1}^{n} z_i e_i\right)\right)\\
&= \theta\left(\sum_{i=1}^{n} y_i e_i, \sum_{i=1}^{n} z_i e_i\right) + \theta\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i\right) + \theta\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} z_i e_i\right)\\
&= \tau\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} z_i\right) + \tau\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) + \tau\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} z_i\right)
\end{aligned}$$

Hence θ is a 2-cocycle of N with values in F.

Proposition 1.3.35. *Let N be the zero algebra of dimension one over a field F with a basis (e_1) and let θ a 2-cocycle on N, as left F-brace. Then there exists an endomorphism τ of $(F,+)$ such that*

$$\theta(x_1e_1, y_1e_1) = \tau(x_1)\, y_1,$$

for all $x_1, y_1 \in F$.

Proof. Let

$$\tau : F \longrightarrow F, \quad x \longmapsto \theta(xe_1, e_1).$$

It follows that

$$\begin{aligned}\tau(x+y) &= \theta((x+y)e_1, e_1) = \theta(xe_1 + ye_1, e_1) = \theta((xe_1) \circ (ye_1), e_1)\\ &= \theta(ye_1, e_1) + \theta(xe_1, (ye_1) \circ e_1) - \theta(xe_1, ye_1)\\ &= \tau(y) + \theta(xe_1, ye_1 + e_1) - \tau(x)\, y = \tau(y) + \tau(x)\, y + \tau(x) - \tau(x)\, y\\ &= \tau(x) + \tau(y).\end{aligned}$$

for all $x, y \in F$, i.e., τ is an endomorphism of the additive group of F. Moreover

$$\theta(x_1e_1, y_1e_1) = y_1\theta(x_1e_1, e_1) = \tau(x_1)\, y_1,$$

for all $x_1, y_1 \in F$. □

This result shows that maps in Example 1.3.34 are the unique 2-cocycles of a 1-dimensional zero algebra.

Proposition 1.3.36. *Let F be a field, B a left F-brace, V a vector space over F and $\theta : B \times B \to V$ a 2-cocycle of the left F-brace B with values in V. If we define on the cartesian product $B \times V$*

$$\begin{aligned}\mu(a, v) &:= (\mu a, \mu v)\\ (a, v) + (b, w) &:= (a + b, v + w)\\ (a, v) \circ (b, w) &:= (a \circ b, v + w + \theta(a, b)),\end{aligned}$$

for all $\mu \in F$, $a, b \in B$ and $v, w \in V$, then $(B \times V, +, \circ)$ is a left F-brace.

Proof. Consider $\tau : B \times B \to V$, defined by $\tau(b_1, b_2) := 0$, for all $b_1, b_2 \in B$. Then τ is a symmetric 2-cocycle from $(B, +)$ with values in V. Moreover if $b_1, b_2, b_3 \in B$, then

$$\begin{aligned}&\theta(b_1, b_2 + b_3) + \tau(b_2, b_3) = \theta(b_1, b_2 + b_3) = \theta(b_1, b_2) + \theta(b_1, b_3)\\&= \theta(b_1, b_2) + \theta(b_1, b_3) - \tau(b_1, -b_1 + b_1 \circ b_3) + \tau(b_1 \circ b_2, -b_1 + b_1 \circ b_3),\end{aligned}$$

i.e., (τ, θ) is a Hochschild pair from the left brace B with values in V with τ symmetric. By Proposition 1.3.28, $B \times V$ respect to the sum and the product given by

$$\begin{aligned}(a, v) + (b, w) &= (a + b, v + w)\\(a, v) \circ (b, w) &= (a \circ b, v + w + \theta(a, b)),\end{aligned}$$

for all $a, b \in B$ and $v, w \in V$ is a left brace. Finally if $\mu \in F$, $a, b \in B$, $v, w \in V$, then, since B is a left F-brace and θ is a 2-cocylce of the left F-brace B with values in V,

$$\begin{aligned}&\mu((a, v) \circ (b, w)) = \mu(a \circ b, v + w + \theta(a, b)) = (\mu(a \circ b), \mu v + \mu w + \mu\theta(a, b))\\&= (a \circ (\mu b) + (\mu - 1)a, \mu v + \mu w + \theta(a, \mu b)) = (a \circ (\mu b), v + \mu w + \theta(a, \mu b))\\&+ ((\mu - 1)a, (\mu - 1)v) = (a, v) \circ (\mu(b, w)) + (\mu - 1)(a, v).\end{aligned}$$

Hence, by Definition 1.1.12, the Hochschild product of B by V (via τ and θ) is a left F-brace. □

Definition 1.3.37. Let F be a field, $(B, +, \circ)$ a left F-brace, V a vector space over F and $\theta : B \times B \to V$ a 2-cocycle of the left F-brace B with values in V. The left F-brace in previous proposition is called a *Hochschild product of the left F-brace B by V (via θ)*.

Theorem 1.3.38. *Let F be a field B, a left F-brace such that $\mathrm{Ann}_F(B) \neq \{0\}$ and $V := \mathrm{Ann}_F(B)$. Then there exists a 2-cocycle θ of the left F-brace $\bar{B} := B/V$ with values in V such that B is isomorphic to the Hochschild product of $\bar{B}$ by V (via θ).*

Proof. As in Theorem 1.3.21, let $V := \mathrm{Ann}_F(B)$, $\bar{B} := B/\mathrm{Ann}_F(B)$, $\pi : B \to \bar{B}$ the projection map and choose a linear map $s : \bar{B} \to B$, from the vector space $\bar{B}$ into the vector space B such that $\pi\left(s\left(\bar{b}\right)\right) = \bar{b}$, for every $\bar{b} \in \bar{B}$. The map $\tau : \bar{B} \times \bar{B} \to V$ is

$$\tau\left(\bar{b}_1, \bar{b}_2\right) := s\left(\bar{b}_1\right) + s\left(\bar{b}_2\right) - s\left(\bar{b}_1 + \bar{b}_2\right) = 0,$$

for all $\bar{b}_1, \bar{b}_2 \in \bar{B}$, since s is a linear map, the map $\theta : \bar{B} \times \bar{B} \to V$ is

$$\begin{aligned}\theta\left(\bar{b}_1, \bar{b}_2\right) &:= \left(s\left(\bar{b}_1 \circ \bar{b}_2\right)\right)^{-} \circ s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right) \\ &= -s\left(\bar{b}_1 \circ \bar{b}_2\right) + s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right),\end{aligned}$$

for all $\bar{b}_1, \bar{b}_2 \in \bar{B}$, the right action $\sigma : \bar{B} \to \operatorname{Aut}(B)$ is

$$\sigma_{\bar{b}} : V \to V, \qquad i \mapsto \left(s\left(\bar{b}\right)\right)^{-} \circ i \circ s\left(\bar{b}\right) = i,$$

for every $\bar{b} \in \bar{B}$, since $\operatorname{Ann}_F(B)$ is contained in the centre of $(B, \circ)$ and the left action $\nu : \bar{B} \to \operatorname{Aut}(B)$ is

$$\nu_{\bar{b}} : I \to I, \qquad i \mapsto -s\left(\bar{b}\right) + s\left(\bar{b}\right) \circ i = i,$$

for every $\bar{b} \in \bar{B}$, since i lies in $\operatorname{Ann}_F(B)$. In particular θ is a 2-cocycle from $\left(\bar{B}, \circ\right)$ with values in V respect to the trivial action, further if $\mu, \nu \in F$ and $\bar{b}_1, \bar{b}_2, \bar{b}_3 \in \bar{B}$, then

$$\begin{aligned}&\mu\theta\left(\bar{b}_1, \bar{b}_2\right) + \nu\theta\left(\bar{b}_1, \bar{b}_3\right) = -\mu s\left(\bar{b}_1 \circ \bar{b}_2\right) + \mu\left(s\left(\bar{b}_1\right) \circ s\left(\bar{b}_2\right)\right) \\ &- \nu s\left(\bar{b}_1 \circ \bar{b}_3\right) + \nu\left(s\left(\bar{b}_1\right) \circ s\left(\bar{b}_3\right)\right) \\ &= -s\left(\mu\left(\bar{b}_1 \circ \bar{b}_2\right)\right) + s\left(\bar{b}_1\right) \circ s\left(\mu\bar{b}_2\right) + (\mu - 1)\, s\left(\bar{b}_1\right) \\ &- s\left(\nu\left(\bar{b}_1 \circ \bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ s\left(\nu\bar{b}_3\right) + (\nu - 1)\, s\left(\bar{b}_1\right) \\ &= -s\left(\bar{b}_1 \circ \left(\mu\bar{b}_2\right)\right) - (\mu - 1)\, s\left(\bar{b}_1\right) + s\left(\bar{b}_1\right) \circ s\left(\mu\bar{b}_2\right) + (\mu - 1)\, s\left(\bar{b}_1\right) \\ &- s\left(\bar{b}_1 \circ \left(\nu\bar{b}_3\right)\right) - (\nu - 1)\, s\left(\bar{b}_1\right) + s\left(\bar{b}_1\right) \circ s\left(\nu\bar{b}_3\right) + (\nu - 1)\, s\left(\bar{b}_1\right) \\ &= -s\left(\bar{b}_1 \circ \left(\mu\bar{b}_2\right)\right) + s\left(\bar{b}_1\right) \circ s\left(\mu\bar{b}_2\right) - s\left(\bar{b}_1 \circ \left(\nu\bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ s\left(\nu\bar{b}_3\right) \\ &+ s\left(\bar{b}_1\right) - s\left(\bar{b}_1\right) = -s\left(\bar{b}_1 \circ \left(\mu\bar{b}_2\right) - \bar{b}_1 + \bar{b}_1 \circ \left(\nu\bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ s\left(\mu\bar{b}_2\right) \\ &- s\left(\bar{b}_1\right) + s\left(\bar{b}_1\right) \circ s\left(\nu\bar{b}_3\right) = -s\left(\bar{b}_1 \circ \left(\mu\bar{b}_2 + \nu\bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ \left(s\left(\mu\bar{b}_2\right) + s\left(\nu\bar{b}_3\right)\right) \\ &= -s\left(\bar{b}_1 \circ \left(\mu\bar{b}_2 + \nu\bar{b}_3\right)\right) + s\left(\bar{b}_1\right) \circ s\left(\mu\bar{b}_2 + \nu\bar{b}_3\right) = \theta\left(\bar{b}_1, \mu\bar{b}_2 + \nu\bar{b}_3\right),\end{aligned}$$

i.e., θ is a 2-cocycle of the left F-brace $\bar{B}$ with values in V. Therefore we may consider the left F-brace Hochschild product of B by V (via θ), by 1.3.36. Finally $\psi : \bar{B} \times A \to B$, defined by $\psi\left(\bar{b}, i\right) = s\left(\bar{b}\right) + i$, for every $\left(\bar{b}, i\right) \in \bar{B} \times V$, is an isomorphism from the left brace Hochschild product of B by V (via θ) into the left brace B. Finally, ψ is linear. In fact, if $\mu \in F$ and $\left(\bar{b}, i\right) \in \bar{B} \times V$, then

$$\mu\psi\left(\bar{b}, i\right) = \mu\left(s\left(\bar{b}\right) + i\right) = s\left(\mu\bar{b}\right) + \mu i = \psi\left(\mu\bar{b}, \mu i\right) = \psi\left(\mu\left(\bar{b}, i\right)\right).$$

Hence ψ is an isomorphism from the left F-brace Hochschild product of B by V (via θ) into the left F-brace B. □

1.4 Bijective 1-cocycles and left semi-braces

As for left braces (developed in [29] and already contained in [16]) and for skew left braces (see [19]), we have a natural one-to-one correspondence between left semi-braces and bijective 1-cocycles (to appear in [8]).

Definition 1.4.1. Let $(G,\cdot)$ be a group, $(B,+)$ a semigroup and $\theta : G \to \mathrm{Aut}(B)$ a homomorphism from the group G into the group of automorphisms of the semigroup B. A function $\delta : G \to B$ is said a 1-*cocycle* from G into B if

$$\delta(g \cdot h) = \delta(g) + \theta(g)(\delta(h)),$$

for all $g, h \in G$.

Proposition 1.4.2. *Let $(B,+)$ be a right group, $(B,\cdot)$ a group, $\theta : B \to \mathrm{Aut}(B)$ a homomorphism from the group $(B,\cdot)$ into the group of automorphisms of $(B,+)$ and $\delta : B \to B$ a bijective 1-cocycle. Then, the operation over B given by*

$$a \circ b := \delta\left(\delta^{-1}(a) \cdot \delta^{-1}(b)\right) = a + \theta\left(\delta^{-1}(a)\right)(b),$$

for all $a, b \in B$, define a structure of group isomorphic to $(B,\cdot)$ such that $(B,+,\circ)$ is a left semi-brace.

Proof. The structure $(B,\circ)$ is a group with identity $0 := \delta(1)$ where 1 is the identity of $(B,\cdot)$. In fact

$$a \circ 0 = \delta\left(\delta^{-1}(a) \cdot 1\right) = a = \delta\left(1 \cdot \delta^{-1}(a)\right) = 0 \circ a,$$

for every $a \in B$, moreover if $a \in B$, setting $b := \delta\left(\left(\delta^{-1}(a)\right)^{-1}\right)$, we have

$$a \circ b = \delta\left(\delta^{-1}(a) \cdot \left(\delta^{-1}(a)\right)^{-1}\right) = \delta(1) = 0$$

and

$$b \circ a = \delta\left(\left(\delta^{-1}(a)\right)^{-1} \cdot \delta^{-1}(a)\right) = \delta(1) = 0.$$

Note that

$$a + b = a \circ \theta\left(\left(\delta^{-1}(a)\right)^{-1}\right)(b), \tag{1.17}$$

for all $a, b \in B$. Furthermore, by previous relation,

$$\begin{aligned} a \circ (b+c) &= a + \theta\left(\delta^{-1}(a)\right)(b+c) = a + \theta\left(\delta^{-1}(a)\right)(b) + \theta\left(\delta^{-1}(a)\right)(c) \\ &= a \circ b + \theta\left(\delta^{-1}(a)\right)(c) = a \circ b + a \circ \left(a^{-} + c\right), \end{aligned}$$

for all $a, b, c \in B$. Therefore, $(B, +, \circ)$ is a left semi-brace. □

Conversely, a left semi-brace determines a bijective 1-cocycle.

Proposition 1.4.3. *Let $(B, +, \circ)$ be a left semi-brace. Then, the function id_B is a bijective 1-cocycle, from the group $(B, \circ)$ into the right group $(B, +)$.*

Proof. By Proposition 1.2.1 we have that $\lambda : B \to \mathrm{Aut}(B), b \mapsto \lambda_b$ is a homomorphism from the group $(B, \circ)$ into the group of automorphisms of $(B, +)$. Moreover, $a + b = a \circ \lambda_{a^-}(b)$, for all $a, b \in B$ and it follows that

$$\mathrm{id}_B(a \circ b) = a \circ b = a \circ \lambda_{a^-}(\lambda_a(b)) = a + \lambda_a(b) = \mathrm{id}_B(a) + \lambda_a(\mathrm{id}_B(b)),$$

and so id_B is a 1-cocycle from the group $(B, \circ)$ into the right group $(B, +)$. □

Theorem 1.4.4. *Let $(B, +)$ be a right group. Then, there exists a group $(B, \circ)$ on B such that $(B, +, \circ)$ is a left semi-brace if and only if there exists a group $(B, \cdot)$ and a bijective 1-cocycle $\delta : B \to B$, from the group $(B, \cdot)$ into the right group $(B, +)$.*

In particular, if $(B, +)$ is a group we obtain, as corollary of previous theorem, the one-to-one correspondence between skew left braces and bijective 1-cocycle, see [19].

Theorem 1.4.5. *Let $(B, +)$ be a group. Then, there exists a group $(B, \circ)$ on B such that $(B, +, \circ)$ is a skew left brace if and only if there exists a group $(B, \cdot)$ and a bijective 1-cocycle $\delta : B \to B$ from the group $(B, \cdot)$ into the group $(B, +)$.*

Moreover if $(B, +)$ is an abelian group, we obtain the analogue of this result for left braces.

Theorem 1.4.6. *Let $(B, +)$ be an abelian group. Then, there exists a group $(B, \circ)$ on B such that $(B, +, \circ)$ is a left brace if and only if there exists a group $(B, \cdot)$ and a bijective 1-cocycle $\delta : B \to B$ from the group $(B, \cdot)$ into the abelian group $(B, +)$.*

1.5 Linear semi-cycle sets

Another algebraic structure strictly linked to left braces is linear cycle set, introduced by Rump in [28]. In [8] we introduce a generalization of this structure, the *linear*

semi-cycle set, and show a natural correspondence between left semi-braces and linear semi-cycle sets.

Definition 1.5.1. Let $(X,+)$ be a right group, $\bullet$ an operation on X. Then, $(X,+,\bullet)$ is said to be a *(right) linear semi-cycle set* if the following hold:

1. $\forall\, a \in X \quad \delta_a : X \to X,\ x \mapsto x \bullet a$ is bijective;
2. $\exists\, 0 \in X \quad 0 \bullet 0 = 0$;
3. $\forall\, a \in X \quad a \bullet a + 0 = a \bullet a$;
4. $\forall\, a,b,c \in X \quad (a+b) \bullet c = a \bullet c + b \bullet c$;
5. $\forall\, a,b,c \in X \quad a \bullet (b+c) = (a \bullet b) \bullet (c \bullet b)$.

Let us note that a (right) linear cycle set (see Definition 1.5.12) is a (right) linear semi-cycle set. In fact, if $(X,+)$ is an abelian group, then clearly 3. in Definition 1.5.1 is satisfied and, by the right distributivity of the sum respect to $\bullet$, $0 \bullet 0 = 0$.

Example 1.5.2. Let X be a group and f an endomorphism of X such that $f^2 = f$. Then, set for all $a,b \in X$, $a + b := bf(a)$ and $a \bullet b := b^{-1}af(b)$, the structure $(X,+,\bullet)$ is a linear semi-cycle set. In fact, $(X,+)$ is a right group. Clearly, the map $\delta_b : X \to X,\ x \mapsto x \bullet b$ is bijective. If 1 is the identity of X, $1 \bullet 1 = 1^{-1}1f(1) = 1$. Moreover,

$$a \bullet a + 1 = a^{-1}af\left(a\right) + 1 = f\left(a\right) + 1 = 1f^2\left(a\right) = f\left(a\right) = a \bullet a.$$

Further,

$$\begin{aligned} a \bullet c + b \bullet c &= c^{-1}af\left(c\right) + c^{-1}bf\left(c\right) = c^{-1}bf\left(c\right) f\left(c^{-1}af\left(c\right)\right) \\ &= c^{-1}bf\left(c\right) f\left(c^{-1}\right) f\left(a\right) f^2\left(c\right) = c^{-1}bf\left(a\right) f\left(c\right) \\ &= c^{-1}\left(a+b\right) f\left(c\right) = \left(a+b\right) \bullet c \end{aligned}$$

and finally

$$\begin{aligned} \left(a \bullet b\right) \bullet \left(c \bullet b\right) &= \left(b^{-1}af\left(b\right)\right) \bullet \left(b^{-1}cf\left(b\right)\right) \\ &= \left(b^{-1}cf\left(b\right)\right)^{-1} \left(b^{-1}af\left(b\right)\right) f\left(b^{-1}cf\left(b\right)\right) \\ &= f\left(b\right)^{-1} c^{-1}b\left(b^{-1}af\left(b\right)\right) f\left(b^{-1}\right) f\left(cf\left(b\right)\right) \\ &= \left(cf\left(b\right)\right)^{-1} af\left(cf\left(b\right)\right) = a \bullet \left(cf\left(b\right)\right) \\ &= a \bullet \left(b+c\right). \end{aligned}$$

Definition 1.5.3. Let $(X, +, \bullet)$ be linear semi-cycle set. Then the operation

$$a \circ b := a + \delta_a^{-1}(b) = a + {}^a b,$$

for all $a, b \in X$, is called the *adjoint operation* of X.

The following lemma contains basic properties of a linear semi-cycle set that are useful tools in the study of this structure.

Lemma 1.5.4. *Let $(X, +, \bullet)$ be a linear semi-cycle set and $E(X)$ the set of idempotents of $(X, +)$. Set ${}^a b := \delta_a^{-1}(b)$, the following hold:*

1. $\forall\ a, b, c \in X \quad {}^a(b + c) = {}^a b + {}^a c$;
2. $\forall\ a, b, c \in X \quad {}^a({}^b c) = {}^{a \circ b} c$;
3. $0 \in E(X)$;
4. $\delta_0 = \mathrm{id}_X$.

Proof. Let $a, b, c \in X$. Then,

$$\delta_a\left({}^a b + {}^a c\right) = \left({}^a b + {}^a c\right) \bullet a = {}^a b \bullet a + {}^a c \bullet a = b + c,$$

and so

$${}^a b + {}^a c = \delta_a^{-1}(b + c) = {}^a(b + c).$$

Moreover,

$${}^a\left({}^b c\right) \bullet (a \circ b) = {}^a\left({}^b c\right) \bullet (a + {}^a b) = \left({}^a\left({}^b c\right) \bullet a\right) \bullet \left({}^a b \bullet a\right) = {}^b c \bullet b = c$$

and so

$${}^a\left({}^b c\right) = \delta_{a \circ b}^{-1}(c) = {}^{a \circ b} c.$$

Moreover, by 2. and 3. in Definition 1.5.1,

$$0 + 0 = 0 \bullet 0 + 0 = 0 \bullet 0 = 0,$$

i.e., $0 \in E(X)$. Now, since

$$\delta_0(a) = a \bullet 0 = a \bullet (0 + 0) = (a \bullet 0) \bullet (0 \bullet 0) = (a \bullet 0) \bullet 0 = \delta_0\left(\delta_0(a)\right),$$

we have $\delta_0(a) = a$, i.e., $\delta_0 = \mathrm{id}_X$. □

Proposition 1.5.5. *Let $(X,+,\bullet)$ be linear semi-cycle set. Then the structure $(X,\circ)$ is a group. For every $a\in X$, we denote the inverse of a by a^{-}.*

Proof. Let $a,b,c\in X$. Then,

$$a\circ(b\circ c)=a+{}^{a}\left(b+{}^{b}c\right)=a+{}^{a}b+{}^{a}\left({}^{b}c\right)=a\circ b+{}^{a\circ b}c=(a\circ b)\circ c,$$

i.e., $\circ$ is associative.
Moreover, 0 is the identity element of $(X,\circ)$. In fact, by $3.$ and $4.$ in Lemma 1.5.4,

$$0\circ a=0+{}^{0}a=0+a=a.$$

Note that by $1.$ in Lemma 1.5.4 and $3.$ in Definition 1.5.1,

$$a={}^{a}(a\bullet a)={}^{a}(a\bullet a+0)={}^{a}(a\bullet a)+{}^{a}0=a+{}^{a}0=a\circ 0,$$

i.e, 0 is identity respect to $\circ$.
Since $(X,+)$ is a right group, there exists $t\in X$ such that $a+t=0$. So, $a\circ(t\bullet a)=a+{}^{a}(t\bullet a)=a+t=0$, i.e., $t\bullet a$ is a right inverse for a respect to $\circ$. Therefore, $(X,\circ)$ is a group. □

Theorem 1.5.6. *Let $(X,+,\bullet)$ be a linear semi-cycle set and $\circ$ its adjoint operation. Then, $(X,+,\circ)$ is a left semi-brace. On the other hand, if $(X,+,\circ)$ is a left semi-brace, set $a\bullet b:=b^{-}\circ(b+a)$, for all $a,b\in X$, then the structure $(X,+,\bullet)$ is a linear semi-cycle set.*

Proof. First, if $(X,+,\bullet)$ is a linear semi-cycle set, note that by previous proposition $(X,\circ)$ is a group. Moreover, let $a,b,c\in X$ and $t\in X$ such that $a+t=0$. Then,

$$\begin{aligned}a\circ b+a\circ\left(a^{-}+c\right)&=a+{}^{a}b+a+{}^{a}(t\bullet a+c)=a+{}^{a}b+a+{}^{a}(t\bullet a)+{}^{a}c\\&=a+{}^{a}b+a+t+{}^{a}c=a+{}^{a}b+0+{}^{a}c\\&=a+{}^{a}(b+c)=a\circ(b+c).\end{aligned}$$

Therefore, X satisfies the condition (1.1) in Definition 1.1.1.
Conversely, if $(X,+,\circ)$ is a left semi-brace, by Proposition 1.2.4, $(X,+)$ is a right group and clearly, for all $b\in X$, $\delta_b=\lambda_{b^-}$ so δ_b is bijective. Let 0 be the identity of $(X,\circ)$. Then,

$$0\bullet 0=\delta_0(0)=\lambda_0(0)=0\circ(0+0)=0.$$

If $a \in X$, then,

$$\begin{aligned} a \bullet a + 0 &= \delta_a(a) + 0 = \lambda_{a^-}(a) + 0 = \lambda_{a^-}(a + \lambda_a(0)) \\ &= \lambda_{a^-}(a \circ 0) = \lambda_{a^-}(a) = a \bullet a. \end{aligned}$$

Moreover, property 4. follows from Proposition 1.2.1.
Finally if $a, b, c \in X$, then,

$$\begin{aligned} a \bullet (b + c) &= \lambda^{-1}_{b+c}(a) = \lambda^{-1}_{b \circ \lambda_{b^-}(c)}(a) = \lambda^{-1}_{\lambda_{b^-}(c)}\left(\lambda_b^{-1}\right)(a) \\ &= \delta_{\delta_b(c)}(\delta_b(a)) = (a \bullet b) \bullet (c \bullet b). \end{aligned}$$

Therefore, $(X, +, \bullet)$ is a linear semi-cycle set. □

1.5.1 Linear skew cycle sets

Here we introduce the concept of linear skew cycle set and we show the connection with skew left braces.

Definition 1.5.7. Let $(X, +)$ be a group and $\bullet$ an operation on X. Then $(X, +, \bullet)$ is said to be a *(right) linear skew cycle set* if

1. $\forall a \in X \qquad \delta_a : X \to X,\ x \mapsto x \bullet a$ is bijective;
2. $\forall a, b, c \in X \qquad (a + b) \bullet c = a \bullet c + b \bullet c$;
3. $\forall a, b, c \in X \qquad a \bullet (b + c) = (a \bullet b) \bullet (c \bullet b)$.

Example 1.5.8. Let $(X, +)$ be a group and $\bullet$ an operation on X such that $a \bullet b := a$, for all $a, b \in X$. Then $(X, +, \bullet)$ is a linear skew cycle set. In fact, clearly $\delta_a(x) = x \bullet a = x = \mathrm{id}_X(x)$ is bijective and if $a, b, c \in X$, then

$$\begin{aligned} (a + b) \bullet c &= a + b = a \bullet c + b \bullet c; \\ a \bullet (b + c) &= a = (a \bullet b) \bullet (c \bullet b). \end{aligned}$$

Proposition 1.5.9. *Let $(X, +, \bullet)$ be a linear skew cycle set. If 0 is the identity of $(X, +)$, then $0 \bullet 0 = 0$.*

Proof. By 2. in Definition 1.5.7, we have

$$0 \bullet 0 = (0 + 0) \bullet 0 = 0 \bullet 0 + 0 \bullet 0,$$

and, since $(X, +)$ is a group, $0 \bullet 0 = 0$. □

Remark 1.5.10. If $(X,+,\bullet)$ is a linear skew cycle set, by previous proposition, $(X,+,\bullet)$ is a linear semi-cycle set. Conversely if $(X,+,\bullet)$ is a linear semi-cycle set with $(X,+)$ a group, then it is a linear skew brace.

By previous remark we obtain the following result as a corollary of the Theorem 1.5.6. Also for linear skew cycle set X we may consider the adjoint operation of X as for linear semi-cycle set, i.e.,

$$a \circ b := a + \delta_a^{-1}(b) = a + {}^a b,$$

for all $a, b \in X$.

Theorem 1.5.11. *Let $(X,+,\bullet)$ be a linear skew cycle set and $\circ$ its adjoint operation. Then $(X,+,\circ)$ is a skew left brace. On the other hand, if $(X,+,\circ)$ is a skew left brace, then setting $a \bullet b := b^- \circ (b+a)$, for all $a, b \in X$, we have that $(X,+,\bullet)$ is a linear skew cycle set.*

Proof. If $(X,+,\bullet)$ is a linear skew cycle set, then by previous remark it is also a linear semi-cycle set and, by Theorem 1.5.6, $(X,+,\circ)$ is a left semi-brace. Further $(X,+)$ is a group and so $(X,+,\circ)$ is a skew left brace. Conversely if $(X,+,\circ)$ is a skew left brace, in particular it is a left semi-brace, then $(X,+,\bullet)$ is a linear semi-cycle set and by previous remark it is a linear skew cycle set. □

1.5.2 Linear cycle sets

Now we recall the definition of linear cycle set.

Definition 1.5.12 (see [29])**.** Let $(X,+)$ be an abelian group and $\bullet$ an operation on X. Then $(X,+,\bullet)$ is said to be a *(right) linear cycle set* if

1. $\forall a \in X \qquad \delta_a : X \to X,\ x \mapsto x \bullet a$ is bijective;
2. $\forall a,b,c \in X \qquad (a+b) \bullet c = a \bullet c + b \bullet c$;
3. $\forall a,b,c \in X \qquad a \bullet (b+c) = (a \bullet b) \bullet (c \bullet b)$.

Example 1.5.13. Let R be an unitary ring and I a left ideal of R contained in $\mathrm{J}(R)$, the Jacobson radical of R. Every element of I is quasiregular, then on I we may consider the following operation

$$a \bullet b := (1+b)^{-1} a,$$

for all $a, b \in I$. Then $(I,+,\bullet)$ is a linear cycle set. In fact, clearly $(I,+)$ is an abelian group. Moreover let $a \in I$. If $x, y \in I$ satisfy $\delta_a(x) = \delta_a(y)$, then $(1+a)^{-1} x =$

$(1+a)^{-1} y$ and so $x = y$, i.e., δ_a is injective. If $y \in I$ then set $x := (1+a)\, y \in I$, $\delta_a(x) = y$. Hence δ_a is a bijection. Further, if $a, b, c \in I$, then

$$(a+b) \bullet c = (1+c)^{-1}(a+b) = (1+c)^{-1} a + (1+c)^{-1} b = a \bullet c + b \bullet c$$

and

$$\begin{aligned}(a \bullet b) \bullet (c \bullet b) &= \left((1+b)^{-1} a\right) \bullet \left((1+b)^{-1} c\right) = \left(1 + (1+b)^{-1} c\right)^{-1} (1+b)^{-1} a \\ &= \left((1+b)\left(1 + (1+b)^{-1} c\right)\right)^{-1} a = (1+b+c)^{-1} a = a \bullet (b+c)\,.\end{aligned}$$

Therefore $(I, +, \bullet)$ is a linear cycle set.

Clearly a linear cycle set is a linear skew cycle set and a left skew brace with additive group abelian is a left brace. Therefore following result holds.

Theorem 1.5.14. *Let $(X, +, \bullet)$ be a linear cycle set and $\circ$ its adjoint operation. Then $(X, +, \circ)$ is a left brace. On the other hand, if $(X, +, \circ)$ is a left brace, then setting $a \bullet b := b^{-} \circ (b + a)$, for all $a, b \in X$, we have that $(X, +, \bullet)$ is a linear cycle set.*

2 The asymmetric product

2.1 The asymmetric product of left semi-braces

We focus on a construction: the asymmetric product of left semi-braces. This construction can be found in [11] and in [9] by Catino, Stefanelli and myself for left semi-braces and for left braces and for left F-braces, respectively. In the asymmetric product, we mix two left semi-braces, using classical tools, to obtain another one. Although in the general case the condition of compatibility is complicated we provide examples of some particular cases that simplify this request.

Let B be a left semi-brace, E the set of idempotents of $(B,+)$, 0 the identity of $(B,\circ)$ and $G := B + 0$. Since $B = G + E$, by Proposition 1.2.5 , every element $b \in B$ may be written in a unique way as $b = g + e$, with $g \in G$ and $e \in E$; we call g the *group part* of b and e the *idempotent part* of b.
In our construction it is essential to be able to distinguish the group and idempotent parts of every element. For this purpose we provide the following lemma.

Lemma 2.1.1. *Let $B = G + E$ be a left semi-brace. If $g, g_1, g_2 \in G$, $e, e_1, e_2 \in E$ then*

1. $(g+e)^- = (g+e)^- \circ e + (g+e)^- \circ g$, *with* $(g+e)^- \circ e \in G$ *and* $(g+e)^- \circ g \in E$;

2. $(g_1+e_1)\circ(g_2+e_2) = (g_1+e_1)\circ g_2 + 0 + \lambda_{(g_1+e_1)}(e_2)$, *with* $(g_1+e_1)\circ g_2 + 0 \in G$ *and* $\lambda_{(g_1+e_1)}(e_2) \in E$.

Proof. 1. If $g+e$ is in $G+E$, then

$$\begin{aligned}(g+e)^- &= (g+e)^- \circ (e+0) = (g+e)^- \circ e + (g+e)^- \circ (g+e+0)\\ &= (g+e)^- \circ e + (g+e)^- \circ g\end{aligned}$$

and $(g+e)^- \circ e = \rho_e(g^-) \in G$, by 2. in Proposition 1.2.11, and $(g+e)^- \circ g = (\lambda_{g^-}(e))^- \in E$, by 1. in Proposition 1.2.11 and 2. in Proposition 1.2.7.
2. If g_1+e_1, g_2+e_2 are in $G+E$, then

$$\begin{aligned}(g_1+e_1)\circ(g_2+e_2) &= (g_1+e_1)\circ g_2 + (g_1+e_1)\circ\left((g_1+e_1)^- + e_2\right)\\ &= (g_1+e_1)\circ g_2 + 0 + \lambda_{(g_1+e_1)}(e_2),\end{aligned}$$

with $(g_1+e_1)\circ g_2 + 0 \in G$, and, by 1. in Proposition 1.2.11, $\lambda_{(g_1+e_1)}(e_2) \in E$. □

Definition 2.1.2. Let $(H,+),(N,+)$ be groups, $\mathfrak{c}: H\times H\to N$ and $\alpha: H\to \mathrm{Aut}(N)$. Set $n^h := \alpha(h)(n)$ for all $h\in H$, $n\in N$, if

1. $\left(n^{h_1}\right)^{h_2} = -\mathfrak{c}(h_1,h_2) + n^{h_1+h_2} + \mathfrak{c}(h_1,h_2)$,
2. $\mathfrak{c}(h_1+h_2,h_3) + \mathfrak{c}(h_1,h_2)^{h_3} = \mathfrak{c}(h_1,h_2+h_3) + \mathfrak{c}(h_2,h_3)$,
3. $\mathfrak{c}(h_1,0) = \mathfrak{c}(0,h_2) = 0$,

hold, for all $n\in N$, $h_1,h_2,h_3\in H$, then the structure over the cartesian product $H\times N$ with the sum given by

$$(h_1,n_1)+(h_2,n_2) := \left(h_1+h_2, \mathfrak{c}(h_1,h_2) + {n_1}^{h_2} + n_2\right)$$

is a group with identity $(0,0)$, known as *Schreier's extension* of N by H (via α and $\mathfrak{c}$) [see Theorem 15.1.1 in 21]. We briefly call the couple $(\alpha,\mathfrak{c})$ a *cocycle* from H into N.

We may extend the definition of cocycle between two groups to one between additive structures of two left semi-braces.

Proposition 2.1.3. *Let $B_1 = H+E$ and $B_2 = N+F$ be left semi-brace, where E is the set of idempotents of $(B_1,+)$, F is the set of idempotents of $(B_2,+)$, 0_{B_1} is the identity of $(B_1,\circ)$, $H := B_1 + 0_{B_1}$, 0_{B_2} is the identity of $(B_2,\circ)$, $N := B_2 + 0_{B_2}$ and $(\alpha,\mathfrak{c})$ a cocycle from H into N. Then the sum over the cartesian product $B_1\times B_2$ given by*

$$\begin{aligned}(h_1+e_1,n_1+f_1)+(h_2+e_2,n_2+f_2)&\\ := \left(h_1+h_2+e_2, \mathfrak{c}(h_1,h_2) + {n_1}^{h_2} + n_2 + f_2\right),&\end{aligned}$$

for all $h_1,h_2\in H$, $n_1,n_2\in N$, $e_1,e_2\in E$ and $f_1,f_2\in F$, defines a structure of right group.

Proof. If $h_1+e_1, h_2+e_2, h_3+e_3\in B_1$ and $n_1+f_1, n_2+f_2, n_3+f_3\in B_2$ then

$$\begin{aligned}&(h_1+e_1,n_1+f_1)+((h_2+e_2,n_2+f_2)+(h_3+e_3,n_3+f_3))\\ &= (h_1+e_1,n_1+f_1)+\left(h_2+h_3+e_3, \mathfrak{c}(h_2,h_3)+{n_2}^{h_3}+n_3+f_3\right)\\ &= \left(h_1+h_2+h_3+e_3, \mathfrak{c}(h_1,h_2+h_3)+{n_1}^{h_2+h_3}+\mathfrak{c}(h_2,h_3)+n_2^{h_3}+n_3+f_3\right)\\ &= \left(h_1+h_2+h_3+e_3, \mathfrak{c}(h_1,h_2+h_3)+\mathfrak{c}(h_2,h_3)+\left(n_1^{h_2}\right)^{h_3}+n_2^{h_3}+n_3+f_3\right)\\ &= \left(h_1+h_2+h_3+e_3, \mathfrak{c}(h_1+h_2,h_3)+\mathfrak{c}(h_1,h_2)^{h_3}+\left(n_1^{h_2}\right)^{h_3}+n_2^{h_3}+n_3+f_3\right)\\ &= \left(h_1+h_2+h_3+e_3, \mathfrak{c}(h_1+h_2,h_3)+\left(\mathfrak{c}(h_1,h_2)+n_1^{h_2}+n_2\right)^{h_3}+n_3+f_3\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(h_1 + h_2 + e_2, \mathfrak{c}\left(h_1, h_2\right) + {n_1}^{h_2} + n_2 + f_2\right) + \left(h_3 + e_3, n_3 + e_3\right) \\
&= \left(\left(h_1 + e_1, n_1 + f_1\right) + \left(h_2 + e_2, n_2 + f_2\right)\right) + \left(h_3 + e_3, n_3 + e_3\right)
\end{aligned}
$$

If $h_1 + e_1, h_2 + e_2, h_3 + e_3 \in B_1$ and $n_1 + f_1, n_2 + f_2, n_3 + f_3 \in B_2$ such that

$$
\left(h_1 + e_1, n_1 + e_1\right) + \left(h_2 + e_2, n_2 + e_2\right) = \left(h_1 + e_1, n_1 + e_1\right) + \left(h_3 + e_3, n_3 + e_3\right),
$$

then

$$
\begin{cases}
h_1 + h_2 + e_2 = h_1 + h_3 + e_3 \\
\mathfrak{c}\left(h_1, h_2\right) + n_1^{h_2} + n_2 + f_2 = \mathfrak{c}\left(h_1, h_3\right) + n_1^{h_3} + n_3 + f_3
\end{cases}
$$

it follows

$$
h_2 = h_3, \quad e_2 = e_3, \quad n_2 = n_3, \quad f_2 = f_3,
$$

i. e., $\left(B_1 \times B_2, +\right)$ is a left cancellative semigroup. Moreover if $h_1 + e_1, h_3 + e_3 \in B_1$ and $n_1 + f_1, n_3 + f_3 \in B_2$, set $h_2 := -h_1 + h_3$, $e_2 := e_3$, $n_2 := -n_1^{-h_1+h_3} - \mathfrak{c}\left(h_1, -h_1 + h_3\right) + n_3$ and $f_2 := f_3$, then

$$
\begin{aligned}
&\left(h_1 + e_1, n_1 + f_1\right) + \left(h_2 + e_2, n_2 + f_2\right) \\
&= \left(h_1 + h_2 + e_2, \mathfrak{c}\left(h_1, h_2\right) + {n_1}^{h_2} + n_2 + f_2\right) \\
&= \left(h_1 + \left(-h_1 + h_3\right) + e_3,\right. \\
&\left.\mathfrak{c}\left(h_1, -h_1 + h_3\right) + {n_1}^{-h_1+h_3} + \left(-n_1^{-h_1+h_3} - \mathfrak{c}\left(h_1, -h_1 + h_3\right) + n_3\right) + f_3\right) \\
&= \left(h_3 + e_3, n_3 + f_3\right).
\end{aligned}
$$

Hence $\left(B_1 \times B_2, +\right)$ is a right group. □

Theorem 2.1.4. *Let $B_1 = H + E, B_2 = N + F$ be left semi-braces, $(\alpha, \mathfrak{c})$ a cocyle from H into N and $\beta : B_2 \to \mathrm{Aut}(B_1)$ a homomorphism from the group $(B_2, \circ)$ into the group of automorphisms of the left semi-brace B_1. Set ${}^{(n+f)}\left(h + e\right) := \beta(n + f)(h + e)$ for all $h + e \in B_1$, $n + f \in B_2$, if*

$$
\begin{aligned}
&\mathfrak{c}\left(\left(h_1 + e_1\right) \circ {}^{(n_1+f_1)}h_2 + 0, \lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}h_3\right) + 0\right) \\
&+ \left(\left(n_1 + f_1\right) \circ n_2 + 0\right)^{\lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}h_3\right)+0} \\
&+ \left(n_1 + f_1\right) \circ \left(\mathfrak{c}\left({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right), h_3\right) + \left(\rho_{f_1}\left(n_1^-\right)\right)^{h_3}\right) \\
&= \left(n_1 + f_1\right) \circ \left(\mathfrak{c}\left(h_2, h_3\right) + {n_2}^{h_3}\right)
\end{aligned}
\tag{2.1}
$$

holds for all $h_1, h_2, h_3 \in H$, $e_1 \in E$, $n_1, n_2 \in N$, $f_1 \in F$, *then the sum and the multiplication over the cartesian product* $B_1 \times B_2$ *given by*

$$\begin{aligned}
&(h_1 + e_1,\ n_1 + f_1) + (h_2 + e_2, n_2 + f_2) \\
&\qquad := \left(h_1 + h_2 + e_2,\ \mathfrak{c}\,(h_1, h_2) + {n_1}^{h_2} + n_2 + f_2\right) \\
&(h_1 + e_1,\ n_1 + f_1) \circ (h_2 + e_2, n_2 + f_2) \\
&\qquad := \left((h_1 + e_1) \circ^{(n_1+f_1)} (h_2 + e_2)\,,\ (n_1 + f_1) \circ (n_2 + f_2)\right)
\end{aligned}$$

define a structure of left semi-brace. We call this structure the asymmetric product of B_1 and B_2 (via α, $\mathfrak{c}$ and β) *and denote it by* $B_1 \rtimes_\circ B_2$.

Proof. By Proposition 2.1.3, $(B_1 \times B_2, +)$ is a right group. Moreover, $(B_1 \times B_2, \circ)$ is a group since it is the semidirect product of the groups $(B_1, \circ)$ and $(B_2, \circ)$ by β. To prove relation (1.1) in Definition 1.1.1, first we note that if $h_1 + e_1, h_2 + e_2 \in B_1$ and $n_1 + f_1, n_2 + f_2 \in B_2$, then

$$\begin{aligned}
&(h_1 + e_1,\ n_1 + f_1) \circ (h_2 + e_2, n_2 + f_2) \\
&= \left((h_1 + e_1) \circ^{(n_1+f_1)} (h_2 + e_2)\,, (n_1 + f_1) \circ (n_2 + f_2)\right) \\
&= \Big((h_1 + e_1) \circ^{(n_1+f_1)} h_2 + 0 + \lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}e_2\right), \\
&\quad (n_1 + f_1) \circ n_2 + 0 + \lambda_{(n_1+f_1)}(f_2)\Big),
\end{aligned}$$

by Lemma 2.1.1. Now, let $(h_1 + e_1, n_1 + f_1), (h_2 + e_2, n_2 + f_2), (h_3 + e_3, n_3 + f_3)$ be in $B_1 \times B_2$. Then we have

$$\begin{aligned}
L :=& (h_1 + e_1, n_1 + f_1) \circ ((h_2 + e_2, n_2 + f_2) + (h_3 + e_3, n_3 + f_3)) \\
&= (h_1 + e_1, n_1 + f_1) \circ \left(h_2 + h_3 + e_3, \mathfrak{c}\,(h_2, h_3) + n_2^{h_3} + n_3 + f_3\right) \\
&= \Big((h_1 + e_1) \circ^{(n_1+f_1)} (h_2 + h_3) + 0 + \lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}e_3\right), \\
&\quad (n_1 + f_1) \circ \left(\mathfrak{c}\,(h_2, h_3) + n_2^{h_3} + n_3\right) + 0 + \lambda_{(n_1+f_1)}(f_3)\Big).
\end{aligned}$$

Moreover, by the Lemma 2.1.1 and by propositions 1.2.7 and 1.2.11,

$$\begin{aligned}
&(h_1 + e_1, n_1 + f_1)^- + (h_3 + e_3, n_3 + f_3) \\
&= \left({}^{(n_1+f_1)^-}(h_1 + e_1)^-, (n_1 + f_1)^-\right) + (h_3 + e_3, n_3 + f_3) \\
&= \Big({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right) + \left(\lambda_{h_1^-}(e_1)\right)^-\right), \\
&\quad \rho_{f_1}\left(n_1^-\right) + \left(\lambda_{n_1^-}(f_1)\right)^-\Big) + (h_3 + e_3, n_3 + f_3) \\
&= \Big({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right) + h_3 + e_3, \\
&\quad \mathfrak{c}\left({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right), h_3\right) + \left(\rho_{f_1}\left(n_1^-\right)\right)^{h_3} + n_3 + f_3\Big).
\end{aligned}$$

This implies that

$$
\begin{aligned}
&(h_1+e_1,n_1+f_1)\circ\left((h_1+e_1,n_1+f_1)^{-}+(h_3+e_3,n_3+f_3)\right)\\
&=\left((h_1+e_1)\circ^{(n_1+f_1)}\left({}^{(n_1+f_1)^{-}}\left(\rho_{e_1}\left(h_1^{-}\right)\right)+h_3\right)+0+\lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}e_3\right),\right.\\
&(n_1+f_1)\circ\left(\mathfrak{c}\left({}^{(n_1+f_1)^{-}}\left(\rho_{e_1}\left(h_1^{-}\right)\right),h_3\right)+\left(\rho_{f_1}\left(n_1^{-}\right)\right)^{h_3}+n_3\right)+\\
&\left.0+\lambda_{(n_1+f_1)}\left(f_3\right)\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&R:=(h_1+e_1,n_1+f_1)\circ(h_2+e_2,n_2+f_2)+(h_1+e_1,n_1+f_1)\\
&\circ\left((h_1+e_1,n_1+f_1)^{-}+(h_3+e_3,n_3+f_3)\right)=\left((h_1+e_1)\circ^{(n_1+f_1)}h_2\right.\\
&+(h_1+e_1)\circ^{(n_1+f_1)}\left({}^{(n_1+f_1)^{-}}\left(\rho_{e_1}\left(h_1^{-}\right)\right)+h_3\right)+0+\lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}e_3\right),\\
&\mathfrak{c}\left((h_1+e_1)\circ^{(n_1+f_1)}h_2+0,(h_1+e_1)\circ^{(n_1+f_1)}\left({}^{(n_1+f_1)^{-}}\left(\rho_{e_1}\left(h_1^{-}\right)\right)+h_3\right)+0\right)\\
&+((n_1+f_1)\circ n_2+0)^{(h_1+e_1)\circ^{(n_1+f_1)}\left({}^{(n_1+f_1)^{-}}\left(\rho_{e_1}\left(h_1^{-}\right)\right)+h_3\right)+0}\\
&+(n_1+f_1)\circ\left(\mathfrak{c}\left({}^{(n_1+f_1)^{-}}\left(\rho_{e_1}\left(h_1^{-}\right)\right),h_3\right)+\left(\rho_{f_1}\left(n_1^{-}\right)\right)^{h_3}+n_3\right)\\
&\left.+0+\lambda_{(n_1+f_1)}\left(f_3\right)\right).
\end{aligned}
$$

Since the idempotent components of the two first projections are the same, we may compare the component in H of the first projection of R with that of L using Lemma 2.1.1,

$$
\begin{aligned}
&(h_1+e_1)\circ^{(n_1+f_1)}h_2+(h_1+e_1)\circ^{(n_1+f_1)}\left({}^{(n_1+f_1)^{-}}\left(\rho_{e_1}\left(h_1^{-}\right)\right)+h_3\right)+0\\
&=(h_1+e_1)\circ^{(n_1+f_1)}h_2+(h_1+e_1)\circ\left(\rho_{e_1}\left(h_1^{-}\right)+{}^{(n_1+f_1)}h_3\right)+0\\
&=(h_1+e_1)\circ^{(n_1+f_1)}h_2+(h_1+e_1)\circ\left(\rho_{e_1}\left(h_1^{-}\right)+\left(\lambda_{h_1^{-}}\left(e_1\right)\right)^{-}+{}^{(n_1+f_1)}h_3\right)+0\\
&=(h_1+e_1)\circ^{(n_1+f_1)}h_2+(h_1+e_1)\circ\left((h_1+e_1)^{-}+{}^{(n_1+f_1)}h_3\right)+0\\
&=(h_1+e_1)\circ\left({}^{(n_1+f_1)}h_2+{}^{(n_1+f_1)}h_3\right)+0=(h_1+e_1)\circ^{(n_1+f_1)}\left(h_2+h_3\right)+0.
\end{aligned}
$$

So the first projections coincide. For the second projections, idempotent components are the same. So we compare components in N. For R the component in N is

$$
\begin{aligned}
&\mathfrak{c}\left((h_1+e_1)\circ{}^{(n_1+f_1)}h_2+0,(h_1+e_1)\circ{}^{(n_1+f_1)}\left({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right)+h_3\right)+0\right)\\
&+((n_1+f_1)\circ n_2+0)^{(h_1+e_1)\circ{}^{(n_1+f_1)}\left({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right)+h_3\right)+0}\\
&+(n_1+f_1)\circ\left(\mathfrak{c}\left({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right),h_3\right)+\left(\rho_{f_1}\left(n_1^-\right)\right)^{h_3}+n_3\right)+0\\
&=\mathfrak{c}\left((h_1+e_1)\circ{}^{(n_1+f_1)}h_2+0,(h_1+e_1)\circ\left(\left(\rho_{e_1}\left(h_1^-\right)\right)+{}^{(n_1+f_1)}h_3\right)+0\right)\\
&+((n_1+f_1)\circ n_2+0)^{(h_1+e_1)\circ\left(\left(\rho_{e_1}\left(h_1^-\right)\right)+{}^{(n_1+f_1)}h_3\right)+0}\\
&+(n_1+f_1)\circ\left(\mathfrak{c}\left({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right),h_3\right)+\left(\rho_{f_1}\left(n_1^-\right)\right)^{h_3}+n_3\right)+0\\
&=\mathfrak{c}\left((h_1+e_1)\circ{}^{(n_1+f_1)}h_2+0,\lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}h_3\right)+0\right)\\
&+((n_1+f_1)\circ n_2+0)^{\lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}h_3\right)+0}\\
&+(n_1+f_1)\circ\left(\mathfrak{c}\left({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right),h_3\right)+\left(\rho_{f_1}\left(n_1^-\right)\right)^{h_3}+n_3\right)+0
\end{aligned}
$$

and by (2.1) we have that it is equal to the component in N of the second projection of L. □

We obtain the semidirect product of two left semi-brace, a generalization of the semidirect product of groups, as particular case of the asymmetric product.

Corollary 2.1.5. *Let B_1 and B_2 be left semi-braces. Let $\beta : B_2 \to \mathrm{Aut}(B_1)$ be a homomorphism from the group $(B_2,\circ)$ into the group of automorphism of the left semi-brace $(B_1,+,\circ)$. Define on $B_1\times B_2$*

$$
\begin{aligned}
&(h_1+e_1,\ n_1+f_1)+(h_2+e_2,n_2+f_2):=(h_1+h_2+e_2,n_1+n_2+f_2)\\
&(h_1+e_1,\ n_1+f_1)\circ(h_2+e_2,n_2+f_2)\\
&\qquad\qquad:=\left((h_1+e_1)\circ{}^{(n_1+f_1)}(h_2+e_2),(n_1+f_1)\circ(n_2+f_2)\right).
\end{aligned}
$$

Then $(B_1\times B_2,+,\circ)$ is a left semi-brace. We call this structure the semidirect product of B_1 and B_2 (via β) *and we denote it by $B_1\rtimes B_2$.*

Proof. Let $B_1 = H + E$ and $B_2 = N + F$, $\alpha : H \to \mathrm{Aut}(N)$ such that $\alpha(h) = \mathrm{id}_N$, for every $h \in H$, and $\mathfrak{c} : H \times H \to N$ such that $\mathfrak{c}(h_1, h_2) = 0$, for all $h_1, h_2 \in H$. Then $(\alpha, \mathfrak{c})$ is a cocycle from H into N and

$$\begin{aligned}
&\mathfrak{c}\left((h_1+e_1)\circ {}^{(n_1+f_1)}h_2 + 0, \lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}h_3\right) + 0\right)\\
&+ ((n_1+f_1)\circ n_2 + 0)^{\lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}h_3\right)+0}\\
&+ (n_1+f_1)\circ\left(\mathfrak{c}\left({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right), h_3\right) + \left(\rho_{f_1}\left(n_1^-\right)\right)^{h_3}\right)\\
&= ((n_1+f_1)\circ n_2 + 0) + (n_1+f_1)\circ\left(\rho_{f_1}\left(n_1^-\right)\right)\\
&= (n_1+f_1)\circ n_2 + (n_1+f_1)\circ\left(\rho_{f_1}\left(n_1^-\right) + 0\right)\\
&= (n_1+f_1)\circ n_2 + (n_1+f_1)\circ\rho_{f_1}\left(n_1^-\right) + (n_1+f_1)\circ\left((n_1+f_1)^- + 0\right)\\
&= (n_1+f_1)\circ n_2 + f_1 + (n_1+f_1)\circ\left((n_1+f_1)^- + 0\right)\\
&= (n_1+f_1)\circ n_2 + (n_1+f_1)\circ\left((n_1+f_1)^- + 0\right)\\
&= (n_1+f_1)\circ(n_2+0) = (n_1+f_1)\circ n_2 = (n_1+f_1)\circ\left(\mathfrak{c}(h_2,h_3) + {n_2}^{h_3}\right)
\end{aligned}$$

holds for all $h_1, h_2, h_3 \in H$, $e_1 \in E$, $n_1, n_2 \in N$ and $f_1 \in F$. Therefore condition (2.1) in Theorem 2.1.4 is satisfied and as a consequence $(B_1 \times B_2, +, \circ)$ is a left semi-brace. □

Example 2.1.6. Let G be a group. Let B_1 be the zero left semi-brace with G as additive (and multiplicative) structure (see Example 1.1.6), B_2 the trivial left semi-brace with G as multiplicative structure (see Example 1.1.3), and $\beta : B_2 \to \mathrm{Aut}(B_1)$ a group homomorphism. The semidirect product of $B_1 \rtimes B_2$ by β is given by

$$(h_1, f_1) + (h_2, f_2) = (h_1 + h_2, f_2)$$

and

$$(h_1, f_1)\circ(h_2, f_2) = (h_1 +^{f_1} h_2, f_2)$$

for all $h_1, h_2 \in B_1$ and $f_1, f_2 \in B_2$.
Let us note that this left semi-brace is not a trivial left semi-brace neither a zero left semi-brace.

Finally, notice that if B_1 is a trivial left semi-brace, B_2 a left semi-brace and β a homomorphism from the group $(B_2, \circ)$ into the group of automorphisms of the group $(B_1, \circ)$, since $\mathrm{Aut}(B_1, \circ) \subseteq \mathrm{Aut}(B_1, +, \circ)$, then we may consider the semidirect product of left semi-braces $B_1 \rtimes B_2$ via β. So the semidirect product over $B_1 \times B_2$ is given by

$$(e_1, n_1 + f_1) + (e_2, n_2 + f_2) = (e_2, n_1 + n_2 + f_2)$$

and

$$(e_1, n_1 + f_1) \circ (e_2, n_2 + f_2) = (e_1 \circ {}^{n_1+f_1}e_2, (n_1 + f_1) \circ (n_2 + f_2)),$$

for all $e_1, e_2 \in B_1$ and $n_1 + f_1, n_2 + f_2 \in B_2$.

2.2 The asymmetric product of skew left braces

Corollary 2.2.1. *Let H, N be skew left braces, $(\alpha, \mathfrak{c})$ a cocycle from H into N and $\beta : N \to \mathrm{Aut}(H)$ a homomorphism from the group $(N, \circ)$ into the group of automorphisms of the skew left brace H such that*

$$\begin{aligned} & n_1 \circ \mathfrak{c}(h_2, h_3) - \left(n_1 \circ (-n_2)^{h_3}\right) \\ & = \mathfrak{c}(h_1 \circ {}^{n_1}h_2, -h_1 + h_1 \circ {}^{n_1}h_3) + (n_1 \circ n_2 - n_1)^{-h_1+h_1\circ{}^{n_1}h_3} \\ & -\mathfrak{c}(h_1, -h_1 + h_1 \circ {}^{n_1}h_3), \end{aligned} \tag{2.2}$$

holds for all $n_1, n_2 \in N$, $h_1, h_2, h_3 \in H$. Then the sum and the multiplication over the cartesian product $H \times N$ respectively given by

$$\begin{aligned} (h_1, n_1) + (h_2, n_2) &:= \left(h_1 + h_2, \mathfrak{c}(h_1, h_2) + {n_1}^{h_2} + n_2\right) \\ (h_1, n_1) \circ (h_2, n_2) &:= (h_1 \circ {}^{n_1}h_2, n_1 \circ n_2) \end{aligned}$$

define a structure of skew left brace, called the asymmetric product of H by N (via α, β and $\mathfrak{c}$) *and denoted by $H \rtimes_\circ N$.*

Proof. Let $h_1, h_2, h_3 \in H$ and $n_1, n_2 \in N$. First, notice that in this case the condition (2.1) becomes

$$\begin{aligned} n_1 \circ \left(\mathfrak{c}(h_2, h_3) + n_2^{h_3}\right) &= \mathfrak{c}(h_1 \circ {}^{n_1}h_2, \lambda_{h_1}({}^{n_1}h_3)) + (n_1 \circ n_2)^{\lambda_{h_1}({}^{n_1}h_3)} \\ &\quad + n_1 \circ \left(\mathfrak{c}\left({}^{n_1^-}h_1^-, h_3\right) + \left(n_1^-\right)^{h_3}\right). \end{aligned}$$

Now, by (2.2) and by 1. in Proposition 1.2.7, we have that

$$\begin{aligned} & n_1 \circ \left(\mathfrak{c}\left({}^{n_1^-}h_1^-, h_3\right) + \left(n_1^-\right)^{h_3}\right) = n_1 \circ \mathfrak{c}\left({}^{n_1^-}h_1^-, h_3\right) + \lambda_{n_1}\left(\left(n_1^-\right)^{h_3}\right) \\ & = \mathfrak{c}\left(h_1 \circ {}^{n_1}\left({}^{n_1^-}h_1^-\right), \lambda_{h_1}({}^{n_1}h_3)\right) + \left(n_1 \circ n_1^- - n_1\right)^{\lambda_{h_1}({}^{n_1}h_3)} - \mathfrak{c}(h_1, \lambda_{h_1}({}^{n_1}h_3)) \\ & + n_1 \circ \left(-\left(n_1^-\right)^{h_3}\right) + \lambda_{n_1}\left(\left(n_1^-\right)^{h_3}\right) = \mathfrak{c}(0, \lambda_{h_1}({}^{n_1}h_3)) + (-n_1)^{\lambda_{h_1}({}^{n_1}h_3)} \\ & - \mathfrak{c}(h_1, \lambda_{h_1}({}^{n_1}h_3)) + n_1 \circ \left(-\left(n_1^-\right)^{h_3} + \left(n_1^-\right)^{h_3}\right) \\ & = (-n_1)^{\lambda_{h_1}({}^{n_1}h_3)} - \mathfrak{c}(h_1, \lambda_{h_1}({}^{n_1}h_3)) + n_1. \end{aligned}$$

Hence, by (2.2),

$$\begin{aligned}
&\mathfrak{c}\left(h_1 \circ {}^{n_1}h_2, \lambda_{h_1}\left({}^{n_1}h_3\right)\right) + \left(n_1 \circ n_2\right)^{\lambda_{h_1}\left({}^{n_1}h_3\right)} + n_1 \circ \left(\mathfrak{c}\left({}^{n_1^-}h_1^-, h_3\right) + \left(n_1^-\right)^{h_3}\right) \\
&= \mathfrak{c}\left(h_1 \circ {}^{n_1}h_2, \lambda_{h_1}\left({}^{n_1}h_3\right)\right) + \left(n_1 \circ n_2\right)^{\lambda_{h_1}\left({}^{n_1}h_3\right)} + \left(-n_1\right)^{\lambda_{h_1}\left({}^{n_1}h_3\right)} \\
&- \mathfrak{c}\left(h_1, \lambda_{h_1}\left({}^{n_1}h_3\right)\right) + n_1 = n_1 \circ \mathfrak{c}\left(h_2, h_3\right) - \left(n_1 \circ \left(-n_2\right)^{h_3}\right) + n_1 \\
&= n_1 \circ \mathfrak{c}\left(h_2, h_3\right) - n_1 + \left(n_1 \circ \left(n_2\right)^{h_3}\right) - n_1 + n_1 \\
&= n_1 \circ \left(\mathfrak{c}\left(h_2, h_3\right) + n_2^{h_3}\right)
\end{aligned}$$

and so the condition (2.1) is satisfied and $H \rtimes_\circ N$ is a left semi-brace by Theorem 2.1.4. Moreover, $(H \times N, +)$ is a group and then $H \rtimes_\circ N$ is a skew left brace. □

Example 2.2.2. Let H, N be a zero left semi-braces with N abelian. If $\mathfrak{c} : H \times H \to N$ is such that $\mathfrak{c}(h_1, h_2) = 0$, for all $h_1, h_2 \in H$, and $\alpha : H \to \mathrm{Aut}(N)$ is a group homomorphism from the group $(H, +)$ into the group of the automorphisms of the group $(N, +)$ and $\beta : N \to \mathrm{Aut}(H)$ is such that $\beta(n) = \mathrm{id}_H$, for every $n \in N$, then $(\alpha, \mathfrak{c})$ is a cocycle from H into N and

$$\begin{aligned}
&\mathfrak{c}\left(h_1 \circ {}^{n_1}h_2, -h_1 + h_1 \circ {}^{n_1}h_3\right) + \left(n_1 \circ n_2 - n_1\right)^{-h_1 + h_1 \circ {}^{n_1}h_3} \\
&-\mathfrak{c}\left(h_1, -h_1 + h_1 \circ {}^{n_1}h_3\right) \\
&= 0 + \left(n_1 + n_2 - n_1\right)^{-h_1 + h_1 + h_3} = n_2^{h_3} = n_1 - n_1 + n_2^{h_3} \\
&= n_1 \circ 0 - \left(n_1 + \left(-n_2\right)^{h_3}\right) \\
&= n_1 \circ \mathfrak{c}\left(h_2, h_3\right) - \left(n_1 \circ \left(-n_2\right)^{h_3}\right).
\end{aligned}$$

Therefore, the hypotheses of previous corollary are satisfied. In particular, the sum and the product over $H \times N$ are given by

$$\begin{aligned}
\left(h_1, n_1\right) + \left(h_2, n_2\right) &:= \left(h_1 + h_2, {n_1}^{h_2} + n_2\right) \\
\left(h_1, n_1\right) \circ \left(h_2, n_2\right) &:= \left(h_1 + h_2, n_1 + n_2\right).
\end{aligned}$$

This is the same example provided in Example 1.5 in [19].

Corollary 2.2.3. *Let H and N be skew left braces. Let $\beta : N \to \mathrm{Aut}(H)$ be a homomorphism from the group $(N, \circ)$ into the group of automorphism of the skew left brace $(H, +, \circ)$. Define on $H \times N$*

$$\begin{aligned}
\left(h_1, n_1\right) + \left(h_2, n_2\right) &:= \left(h_1 + h_2, n_1 + n_2\right) \\
\left(h_1, n_1\right) \circ \left(h_2, n_2\right) &:= \left(h_1 \circ {}^{n_1}h_2, n_1 \circ n_2\right).
\end{aligned}$$

Then $(H \times N, +, \circ)$ is a skew left brace. We call this structure the semidirect product *of H and N (via β) and we denote it by $H \rtimes N$.*

Proof. As in Corollary 2.1.5, let $\alpha : H \to \mathrm{Aut}(N)$ such that $\alpha(h) = \mathrm{id}_N$, for every $h \in H$ and $\mathfrak{c} : H \times H \to N$ such that $\mathfrak{c}(h_1, h_2) = 0$, for all $h_1, h_2 \in H$. Then $(\alpha, \mathfrak{c})$ is a cocycle from H into N. Moreover, note that if B is a skew left brace, then

$$a = a \circ 0 = a \circ (b - b) = a \circ b - a + a \circ (-b),$$

for all $a, b \in B$, i.e., $-a \circ (-b) = -a + a \circ b - a$, for all $a, b \in B$. Hence

$$\begin{aligned} &n_1 \circ \mathfrak{c}(h_2, h_3) - \left(n_1 \circ (-n_2)^{h_3}\right) = n_1 \circ 0 - n_1 + n_1 \circ n_2 - n_1 \\ &= n_1 - n_1 + n_1 \circ n_2 - n_1 = n_1 \circ n_2 - n_1 \\ &= \mathfrak{c}(h_1 \circ {}^{n_1}h_2, -h_1 + h_1 \circ {}^{n_1}h_3) + (n_1 \circ n_2 - n_1)^{-h_1 + h_1 \circ {}^{n_1}h_3} - \mathfrak{c}(h_1, -h_1 + h_1 \circ {}^{n_1}h_3) \end{aligned}$$

holds for all $h_1, h_2, h_3 \in H$ and $n_1, n_2 \in N$. Therefore the condition (2.2) in Corollary 2.2.1 is satisfied and so $(H \times N, +, \circ)$ is a skew left brace. □

2.3 The asymmetric product of left braces

Remark 2.3.1. Let H, N be abelian groups, $\mathfrak{c} : H \times H \to N$ a symmetric cocycle from H with values in N (see Section 1.3.1). Note that if we consider $\alpha : H \to \mathrm{Aut}(N)$, such that $\alpha(h) = \mathrm{id}_N$, for every $h \in H$, then $(\alpha, \mathfrak{c})$ is a cocycle from H into N as in Definition 2.1.2.

Corollary 2.3.2. *Let H, N be left braces, $\mathfrak{c}$ a symmetric cocycle from H into N and $\beta : N \to \mathrm{Aut}(H)$ a homomorphism from the group $(N, \circ)$ into the group of automorphisms of the left brace H such that*

$$n \circ \mathfrak{c}(h_2, h_3) + \mathfrak{c}(h_1 \circ {}^{n}(h_2 + h_3), h_1) = \mathfrak{c}(h_1 \circ {}^{n}h_2, h_1 \circ {}^{n}h_3) + n \tag{2.3}$$

holds for all $h_1, h_2, h_3 \in H$ and $n \in N$. Then the sum and the multiplication over the cartesian product $H \times N$ respectively given by

$$\begin{aligned} (h_1, n_1) + (h_2, n_2) &:= (h_1 + h_2, \mathfrak{c}(h_1, h_2) + n_1 + n_2) \\ (h_1, n_1) \circ (h_2, n_2) &:= (h_1 \circ {}^{n_1}h_2, n_1 \circ n_2) \end{aligned}$$

define a structure of left brace, called the asymmetric product of H by N (via α and $\mathfrak{c}$) *and denoted by $H \rtimes_{\circ} N$.*

Proof. Let $\alpha : H \to \operatorname{Aut}(N)$, such that $\alpha(h) = \mathrm{id}_N$, for every $h \in H$, then $(\alpha, \mathfrak{c})$ is a cocycle from H into N as in Remark 2.3.1. Moreover if $n_1, n_2 \in N$ and $h_1, h_2, h_3 \in H$, by (2.3), then

$$\begin{aligned}
&\mathfrak{c}(h_1 \circ {}^{n_1}h_2, -h_1 + h_1 \circ {}^{n_1}h_3) + n_1 \circ n_2 - n_1 - \mathfrak{c}(h_1, -h_1 + h_1 \circ {}^{n_1}h_3) \\
&= -\mathfrak{c}(h_1 \circ {}^{n_1}h_2 - h_1 + h_1 \circ {}^{n_1}h_3, h_1) + \mathfrak{c}(h_1 \circ {}^{n_1}h_2, h_1 \circ {}^{n_1}h_3) + n_1 \circ n_2 - n_1 \\
&= -\mathfrak{c}(h_1 \circ {}^{n_1}(h_2 + h_3), h_1) + \mathfrak{c}(h_1 \circ {}^{n_1}h_2, h_1 \circ {}^{n_1}h_3) + n_1 \circ n_2 - n_1 \\
&= n_1 \circ \mathfrak{c}(h_2, h_3) - n_1 + n_1 \circ n_2 - n_1 \\
&= n_1 \circ \mathfrak{c}(h_2, h_3) - (n_1 \circ (-n_2)),
\end{aligned}$$

i.e., the condition (2.2) is satisfied. Hence, by Corollary 2.2.1, $H \rtimes_\circ N$ is a skew left brace. Moreover, if $h_1, h_2 \in H$ and $n_1, n_2 \in N$, then

$$\begin{aligned}
(h_1, n_1) + (h_2, n_2) &= (h_1 + h_2, \mathfrak{c}(h_1, h_2) + n_1 + n_2) \\
&= (h_2 + h_1, \mathfrak{c}(h_2, h_1) + n_2 + n_1) \\
&= (h_2, n_2) + (h_1, n_1),
\end{aligned}$$

i.e., $(H \times N, +)$ is an abelian group. Therefore $H \rtimes_\circ N$ is a left brace. □

Example 2.3.3. Let H be the left brace with additive group $(\mathbb{Z}, +)$ in Example 1.1.10 and $N := \mathbb{Z}/\mathbb{Z}2$ the zero left brace with additive and multiplicative structure the cyclic group of 2 elements. We may consider $\beta : N \to \operatorname{Aut}(H)$ such that $\beta(0) = \mathrm{id}_H$ and $\beta(1) = -\mathrm{id}_H$ and $\mathfrak{c} : H \times H \to N$, such that $\mathfrak{c}(h_1, h_2) = h_1 h_2$. If $h_1, h_2, h_3 \in H$ and $n \in N$, then

$$\begin{aligned}
\mathfrak{c}(h_1 \circ {}^{n}h_2, h_1 \circ {}^{n}h_3) + n &= \mathfrak{c}\left(h_1 + (-1)^{h_1+n} h_2, h_1 + (-1)^{h_1+n} h_3\right) + n \\
&= \left(h_1 + (-1)^{h_1+n} h_2\right)\left(h_1 + (-1)^{h_1+n} h_3\right) + n \\
&= h_1^2 + h_2 h_3 + (-1)^{h_1+n}(h_2 + h_3) h_1 + n \\
&= n + h_2 h_3 + \left(h_1 + (-1)^{h_1+n}(h_2 + h_3)\right) h_1 \\
&= n \circ \mathfrak{c}(h_2, h_3) + \mathfrak{c}(h_1 \circ {}^{n}(h_2 + h_3), h_1),
\end{aligned}$$

i.e., condition (2.3) holds. Therefore $H, N, \beta, \mathfrak{c}$ satisfy the assumption of Corollary 2.3.2. In particular, the sum and product over $H \times N$ are given by

$$\begin{aligned}
(h_1, n_1) + (h_2, n_2) &= (h_1 + h_2, h_1 h_2 + n_1 + n_2), \\
(h_1, n_1) \circ (h_2, n_2) &= \left(h_1 + (-1)^{h_1+n_1} h_2, n_1 + n_2\right),
\end{aligned}$$

for all $h_1, h_2 \in H$ and $n_1, n_2 \in N$.

Example 2.3.4. Let H be the left brace with additive group $(\mathbb{Z}/\mathbb{Z}6, +)$ in Example 1.1.11, $N := \mathbb{Z}/\mathbb{Z}2$ the zero left brace with additive and multiplicative structure the cyclic group of 2 elements. We may consider $\beta : N \to \operatorname{Aut}(H)$ such that

$\beta(0) = \mathrm{id}_H$, $\beta(1) = (1\,5)(2\,4)$ and $\mathfrak{c} : H \times H \to N$, such that $\mathfrak{c}(h_1, h_2) = h_1 * h_2$, where $h_1 * h_2$ is the canonical image of $h_1 h_2$ in $\mathbb{F}_2$. Then, clearly, β is a group homomorphism and $\mathfrak{c}$ is a symmetric cocycle, in fact, $\mathfrak{c}$ is a symmetric map, $\mathfrak{c}(0,0) = 0$ and

$$\begin{aligned}\mathfrak{c}(h_1 + h_2, h_3) + \mathfrak{c}(h_1, h_2) &= (h_1 + h_2) * h_3 + h_1 * h_2 = h_1 * h_3 + h_2 * h_3 + h_1 * h_2 \\ &= h_1 * (h_2 + h_3) + h_2 * h_3 = \mathfrak{c}(h_1, h_2 + h_3) + \mathfrak{c}(h_2, h_3),\end{aligned}$$

for all $h_1, h_2, h_3 \in H$. Moreover, as in the previous example, (2.3) holds. Therefore $H, N, \mathfrak{c}, \beta$ satisfy the assumptions of the Corollary 2.3.2 and we may construct the asymmetric product of H by N. Explicitly the sum and the product over $H \times N$ are given by

$$(h_1, n_1) + (h_2, n_2) = (h_1 + h_2, h_1 * h_2 + n_1 + n_2)$$

and

$$(h_1, n_1) \circ (h_2, n_2) = \begin{cases} (h_1 + h_2, n_1 + n_2) & \text{if } n_1 = 0 \text{ and } h_1 \in \{0, 2, 4\} \\ & \text{or } n_1 = 1 \text{ and } h_1 \in \{1, 3, 5\} \\ (h_1 + 5h_2, n_1 + n_2) & \text{if } n_1 = 0 \text{ and } h_1 \in \{1, 3, 5\} \\ & \text{or } n_1 = 1 \text{ and } h_1 \in \{0, 2, 4\} \end{cases}$$

for all $h_1, h_2 \in H$ and $n_1, n_2 \in N$.

Remark 2.3.5. Note that $\mathbb{Z}6 \times 0$ is an ideal of the left brace $H \rtimes_\circ N$ in Example 2.3.3. In fact, $(\mathbb{Z}6, 0)$ is a normal subgroup of $(H \times N, \circ)$ and

$$\begin{aligned}\lambda_{(h,n)}(6x, 0) &= -(h, n) + (h, n) \circ (6x, 0) = \left(-h, h^2 - n\right) + \left(h + (-1)^{h+n} 6x, n\right) \\ &= \left(-h + h + (-1)^{h+n} 6x, -h\left(h + (-1)^{h+n} 6x\right) + h^2 - n + n\right) \\ &= \left((-1)^{h+n} 6x, 0\right) \in \mathbb{Z}6 \times 0,\end{aligned}$$

for all $h, x \in \mathbb{Z}$ and $n \in \mathbb{Z}/\mathbb{Z}2$. Therefore the left brace in Example 2.3.4 is the quotient of $H \rtimes_\circ N$ by this ideal.

Example 2.3.6. Let $H := \mathbb{Z}/\mathbb{Z}p^2$ be the left brace with multiplication given by

$$h_1 \circ h_2 = h_1 + h_2 + p h_1 h_2,$$

for all $h_1, h_2 \in H$ and $N := \mathbb{Z}/\mathbb{Z}p$ the zero left brace over the cyclic group of p elements. We may consider $\beta : N \to \mathrm{Aut}(H)$ defined by ${}^n h = (1 + pn)\, h$ and

$\mathfrak{c} : H \times H \to N$ such that $\mathfrak{c}(h_1, h_2) = h_1 * h_2$, where $h_1 * h_2$ is the canonical image of $h_1 h_2$ in $\mathbb{F}_p$. Then β is a group homomorphism, $\mathfrak{c}$ is a symmetric cocycle and

$$\begin{aligned}\mathfrak{c}(h_1 \circ^n h_2, h_2 \circ^n h_3) + n =& (h_1 + (1+pn) h_2 + p h_1 (1+pn) h_2) * (h_1 + (1+pn) h_3 \\ &+ p h_1 (1+pn) h_3) + n \\ =& (h_1 + h_2) * (h_1 + h_3) + n \\ =& (h_1 + h_2 + h_3) * h_1 + h_2 * h_3 + n \\ =& n + h_2 * h_3 + (h_1 + (1+pn)(h_2 + h_3) \\ &+ p h_1 (1+pn)(h_2 + h_3)) * h_1 \\ =& n \circ \mathfrak{c}(h_2, h_3) + \mathfrak{c}(h_1 \circ^n (h_2 + h_3), h_1),\end{aligned}$$

for all $h_1, h_2, h_3 \in H$ and $n \in N$. Then $H, N, \beta, \mathfrak{c}$ satisfy the assumption of Corollary 2.3.2, explicitly the sum and the multiplication over $H \times N$ are given by

$$\begin{aligned}(h_1, n_1) + (h_2, n_2) &= (h_1 + h_2, h_1 * h_2 + n_1 + n_2) \\ (h_1, n_1) \circ (h_2, n_2) &= (h_1 + (1 + pn_1) h_2 + p h_1 h_2, n_1 + n_2),\end{aligned}$$

for all $h_1, h_2 \in H$, $n_1, n_2 \in N$. Note that this left brace can be found in [1].

As consequence of the asymmetric product for left braces we find the semidirect product of left braces, introduced in [30], as reformulated in [13].

Corollary 2.3.7. *Let H and N be left braces. Let $\beta : N \to \mathrm{Aut}(H)$ be a homomorphism from the group $(N, \circ)$ into the group of automorphism of the left brace $(H, +, \circ)$. Define on $H \times N$*

$$\begin{aligned}(h_1, n_1) + (h_2, n_2) &:= (h_1 + h_2, n_1 + n_2) \\ (h_1, n_1) \circ (h_2, n_2) &:= (h_1 \circ^{n_1} h_2, n_1 \circ n_2).\end{aligned}$$

Then $(H \times N, +, \circ)$ is a left brace. We call this structure the semidirect product of H and N (via β) *and we denote it by $H \rtimes N$.*

Proof. As in Corollary 2.2.3, let $\alpha : H \to \mathrm{Aut}(N)$ such that $\alpha(h) = \mathrm{id}_N$, for every $h \in H$ and $\mathfrak{c} : H \times H \to N$ such that $\mathfrak{c}(h_1, h_2) = 0$, for all $h_1, h_2 \in H$. Then $(\alpha, \mathfrak{c})$ is a cocycle from H into N. Further

$$n \circ \mathfrak{c}(h_2, h_3) + \mathfrak{c}(h_1 \circ^n (h_2 + h_3), h_1) = \mathfrak{c}(h_1 \circ^n h_2, h_1 \circ^n h_3) + n$$

holds for all $h_1, h_2, h_3 \in H$ and $n \in N$. Therefore the condition (2.3) in Corollary 2.3.2 is satisfied and so $(H \times N, +, \circ)$ is a left brace. □

2.4 The asymmetric product of left F-braces

The following corollary gives a condition to obtain a left F-brace as asymmetric product of left F-braces.

Corollary 2.4.1. *Let F be a field of characteristic $p \neq 2$, and H, N left F-braces. Let $\mathfrak{b} : H \times H \to N$ be bilinear and symmetric map, $\beta : N \to \mathrm{Aut}(H)$ a homomorphism of the multiplicative group of N into the group of the automorphism of the left F-brace H that satisfy the condition*

$$n \circ \mathfrak{b}(h_2, h_3) + \mathfrak{b}(h_1 \circ^n (h_2 + h_3), h_1) = \mathfrak{b}(h_1 \circ^n h_2, h_1 \circ^n h_3) + n, \tag{2.4}$$

for all $h_1, h_2, h_3 \in H$ and $n \in N$. Then the asymmetric product of H by N is a left F-brace with the scalar multiplication given by

$$\mu(h, n) = \left(\mu h, \frac{\mu(\mu-1)}{2}\mathfrak{b}(h, h) + \mu n\right), \tag{2.5}$$

for all $h \in H$, $n \in N$ and $\mu \in F$.

Proof. By Corollary 2.3.2, $H \rtimes_\circ N$ is a left brace. Further $(H \times N, +)$ equipped with the scalar multiplication in (2.5) is a F-vector space. Finally, the scalar multiplication is compatible with $\circ$. In fact, if $\mu \in F$, $h_1, h_2 \in H$, $n_1, n_2 \in N$, then

$$\begin{aligned}
&(h_1, n_1) \circ (\mu(h_2, n_2)) + (\mu - 1)(h_1, n_1) \\
&= (h_1, n_1) \circ \left(\mu h_2, \frac{\mu(\mu-1)}{2}\mathfrak{b}(h_2, h_2) + \mu n_2\right) \\
&+ \left((\mu-1)h_1, \frac{(\mu-1)(\mu-2)}{2}\mathfrak{b}(h_1, h_1) + (\mu-1)n_1\right) \\
&= \left(h_1 \circ^{n_1} (\mu h_2), n_1 \circ \left(\frac{\mu(\mu-1)}{2}\mathfrak{b}(h_2, h_2) + \mu n_2\right)\right) \\
&+ \left((\mu-1)h_1, \frac{(\mu-1)(\mu-2)}{2}\mathfrak{b}(h_1, h_1) + (\mu-1)n_1\right) \\
&= (h_1 \circ^{n_1} (\mu h_2) + (\mu-1)h_1, \mathfrak{b}(h_1 \circ^{n_1} (\mu h_2), (\mu-1)h_1) \\
&+ n_1 \circ \left(\frac{\mu(\mu-1)}{2}\mathfrak{b}(h_2, h_2) + \mu n_2\right) + \frac{(\mu-1)(\mu-2)}{2}\mathfrak{b}(h_1, h_1) + (\mu-1)n_1\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\mu((h_1, n_1) \circ (h_2, n_2)) = \mu(h_1 \circ^{n_1} h_2, n_1 \circ n_2) \\
&= \left(\mu(h_1 \circ^{n_1} h_2), \frac{\mu(\mu-1)}{2}\mathfrak{b}(h_1 \circ^{n_1} h_2, h_1 \circ^{n_1} h_2) + \mu(n_1 \circ n_2)\right)
\end{aligned}$$

since H is a left F-brace, the first components are the same, also second components are the same in fact we have

$$\begin{aligned}
&\mathfrak{b}\left(h_1 \circ^{n_1} (\mu h_2), (\mu-1) h_1\right) + n_1 \circ \left(\frac{\mu(\mu-1)}{2}\mathfrak{b}(h_2, h_2) + \mu n_2\right) \\
&+ \frac{(\mu-1)(\mu-2)}{2}\mathfrak{b}(h_1, h_1) + (\mu-1) n_1 \\
&= \mathfrak{b}\left(\mu(h_1 \circ^{n_1} h_2) - (\mu-1) h_1, (\mu-1) h_1\right) + n_1 \circ \left(\frac{\mu(\mu-1)}{2}\mathfrak{b}(h_2, h_2)\right) \\
&- n_1 + n_1 \circ (\mu n_2) + \frac{(\mu-1)(\mu-2)}{2}\mathfrak{b}(h_1, h_1) + (\mu-1) n_1 \\
&= \mu(\mu-1)\mathfrak{b}\left(h_1 \circ^{n_1} h_2, h_1\right) - (\mu-1)^2 \mathfrak{b}(h_1, h_1) + \frac{\mu(\mu-1)}{2}\left(n_1 \circ \mathfrak{b}(h_2, h_2)\right) \\
&- \left(\frac{\mu(\mu-1)}{2} - 1\right) n_1 - n_1 + \frac{(\mu-1)(\mu-2)}{2}\mathfrak{b}(h_1, h_1) + \mu(n_1 \circ n_2) \\
&= \mu(\mu-1)\mathfrak{b}\left(h_1 \circ^{n_1} h_2, h_1\right) - (\mu-1)^2 \mathfrak{b}(h_1, h_1) \\
&+ \frac{\mu(\mu-1)}{2}\left(\mathfrak{b}\left(h_1 \circ^{n_1} h_2, h_1 \circ^{n_1} h_2\right) + n_1 - \mathfrak{b}\left(h_1 \circ^{n_1} (h_2 + h_2), h_1\right)\right) \\
&- \frac{\mu(\mu-1)}{2} n_1 + \frac{(\mu-1)(\mu-2)}{2}\mathfrak{b}(h_1, h_1) + \mu(n_1 \circ n_2) \\
&= \frac{\mu(\mu-1)}{2}\mathfrak{b}\left(h_1 \circ^{n_1} h_2, h_1 \circ^{n_1} h_2\right) + \mu(n_1 \circ n_2)
\end{aligned}$$

Therefore $H \rtimes_{\circ} N$ is a left F-brace. □

In order to have that (2.5) in previous result is a scalar multiplication the bilinearity of $\mathfrak{b}$ is necessary as the following example shows.

Example 2.4.2. Let $\mathbb{F}_9 := \langle 1 \rangle \oplus \langle \omega \rangle$ be the field with 9 elements, where ω is a root of the irreducible polynomial $x^2 + 2x + 2 \in \mathbb{F}_3[x]$, and $H = N := \mathbb{F}_9$ the zero left $\mathbb{F}_9$-braces. Let $\beta : N \to \mathrm{Aut}(H)$ be the trivial action and $\mathfrak{b} : H \times H \to N$ defined by

$$\mathfrak{b}(a_1 + b_1\omega, a_2 + b_2\omega) = a_1a_2 + b_1b_2 + 2\left((a_1 + b_1) b_2 + a_2b_1\right)\omega,$$

for all $a_1, a_2, b_1, b_2 \in \mathbb{F}_3$. The map $\mathfrak{b}$ is a symmetric cocycle from H into N that satisfies (2.3) but it is not a bilinear map. In fact, if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{F}_3$, then $\mathfrak{b}(0, 0) = 0$,

$$\begin{aligned}
\mathfrak{b}(a_1 + b_1\omega, a_2 + b_2\omega) &= a_1a_2 + b_1b_2 + 2\left((a_1 + b_1) b_2 + a_2b_1\right)\omega \\
&= a_2a_1 + b_2b_1 + 2\left(a_2b_1 + b_2b_1 + a_1b_2\right)\omega \\
&= \mathfrak{b}(a_2 + b_2\omega, a_1 + b_1\omega),
\end{aligned}$$

$$
\begin{aligned}
&\mathfrak{b}\left(\left(a_1+b_1\omega\right)+\left(a_2+b_2\omega\right), a_3+b_3\omega\right)+\mathfrak{b}\left(a_1+b_1\omega, a_2+b_2\omega\right)\\
&=\mathfrak{b}\left(a_1+a_2+\left(b_1+b_2\right)\omega, a_3+b_3\omega\right)+a_1a_2+b_1b_2+2\left(\left(a_1+b_1\right)b_2+a_2b_1\right)\omega\\
&=\left(a_1+a_2\right)a_3+\left(b_1+b_2\right)b_3+2\left(\left(a_1+a_2+b_1+b_2\right)b_3+a_3\left(b_1+b_2\right)\right)\omega\\
&+a_1a_2+b_1b_2+2\left(\left(a_1+b_1\right)b_2+a_2b_1\right)\omega\\
&=a_1a_2+a_1a_3+b_1b_2+b_1b_3+a_2a_3+b_2b_3\\
&+2\left(a_1b_2+a_1b_3+b_1b_2+b_1b_3+a_2b_1+a_3b_1+a_2b_3+b_2b_3+a_3b_2\right)\omega\\
&=a_1\left(a_2+a_3\right)+b_1\left(b_2+b_3\right)+2\left(\left(a_1+b_1\right)\left(b_2+b_3\right)+\left(a_2+a_3\right)b_1\right)\omega\\
&+a_2a_3+b_2b_3+2\left(\left(a_2+b_2\right)b_3+a_3b_2\right)\omega\\
&=\mathfrak{b}\left(a_1+b_1\omega, a_2+a_3+\left(b_2+b_3\right)\omega\right)+\mathfrak{b}\left(a_2+b_2\omega, a_3+b_3\omega\right)\\
&=\mathfrak{b}\left(a_1+b_1\omega,\ \left(a_2+b_2\omega\right)+\left(a_3+b_3\omega\right)\right)+\mathfrak{b}\left(a_2+b_2\omega,\ a_3+b_3\omega\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathfrak{b}\left(a_2+b_2\omega, a_3+b_3\omega\right)+\mathfrak{b}\left(a_1+b_1\omega+a_2+b_2\omega+a_3+b_3\omega, a_1+b_1\omega\right)\\
&=a_2a_3+b_2b_3+2\left(\left(a_2+b_2\right)b_3+a_2b_3\right)\omega+\left(a_1+a_2+a_3\right)a_1+\left(b_1+b_2+b_3\right)b_1\\
&+2\left(\left(a_1+a_2+a_3+b_1+b_2+b_3\right)b_1+a_1\left(b_1+b_2+b_3\right)\right)\omega\\
&=\left(a_1+a_2\right)\left(a_1+a_3\right)+\left(b_1+b_2\right)\left(b_1+b_3\right)\\
&+2\left(\left(a_1+a_2+b_1+b_2\right)\left(b_1+b_3\right)+\left(a_1+a_3\right)\left(b_1+b_2\right)\right)\omega\\
&=\mathfrak{b}\left(a_1+b_1\omega+a_2+b_2\omega, a_1+b_1\omega+a_3+b_3\omega\right).
\end{aligned}
$$

Hence, by Corollary 2.3.2, $H \rtimes_\circ N$ is a left brace. Suppose that (2.5) is a scalar multiplication then

$$
\left(1+\omega\right)\left(2+\omega, 1\right)=\left(\omega, 0\right)
$$

and

$$
\left(2+\omega, 1\right)+\omega\left(2+\omega, 1\right)=\left(\omega, 1+\omega\right),
$$

a contradiction. Therefore $(H \times N, +)$ is not a $\mathbb{F}_9$-vector space.

Moreover, if the characteristic of the field F is 2, then conditions in Corollary 2.4.1 are not sufficient to obtain an F-brace as the following example shows.

Example 2.4.3. Let $\mathbb{F}_2$ be the field with 2 elements, $H := \mathbb{F}_2 \times \mathbb{F}_2$ and $N := \mathbb{F}_2$ the 2-dimensional and 1-dimensional $\mathbb{F}_2$-vector space seen as left $\mathbb{F}_2$-braces. Set $\mathfrak{b} : H \times H \to N$ such that

$$
\mathfrak{b}\left(\left(b_1, c_1\right), \left(b_2, c_2\right)\right)=b_1b_2+c_1c_2,
$$

for all $b_1, b_2, c_1, c_2 \in \mathbb{F}_2$ and $\beta : N \to \mathrm{Aut}(H)$ such that $\beta(0) = \mathrm{id}_H$ and $\beta(1) : H \to H$ such that $\beta(1)(b_1, c_1) = (c_1, b_1)$, for all $b_1, c_1 \in \mathbb{F}_2$. The map $\mathfrak{b}$ is bilinear and if $b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{F}_2$, then

$$\begin{aligned}
&\mathfrak{b}\left((b_1, c_1) + {}^0(b_2, c_2), (b_1, c_1) + {}^0(b_3, c_3)\right) = (b_1 + b_2)(b_1 + b_3) + (c_1 + c_2)(c_1 + c_3) \\
&= b_2 b_3 + c_2 c_3 + (b_1 + b_2 + b_3) b_1 + (c_1 + c_2 + c_3) c_1 \\
&= \mathfrak{b}((b_2, c_2), (b_3, c_3)) + \mathfrak{b}\left((b_1, c_1) + {}^0((b_2, c_2) + (b_3, c_3)), (b_1, c_1)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{b}\left((b_1, c_1) + {}^1(b_2, c_2), (b_1, c_1) + {}^1(b_3, c_3)\right) + 1 \\
&= (b_1 + c_2)(b_1 + c_3) + (c_1 + b_2)(c_1 + b_3) + 1 \\
&= 1 + b_2 b_3 + c_2 1 c_3 + (b_1 + c_2 + c_3) b_1 + (c_1 + b_2 + b_3) c_1 \\
&= 1 + \mathfrak{b}((b_2, c_2), (b_3, c_3)) + \mathfrak{b}\left((b_1, c_1) + {}^1((b_2, c_2) + (b_3, c_3)), (b_1, c_1)\right).
\end{aligned}$$

Hence $H, N, \mathfrak{b}, \beta$ satisfy the assumptions of Corollary 2.3.2. Nevertheless the asymmetric product of H by N is not a left $\mathbb{F}_2$-brace, in fact

$$\begin{aligned}
((1,0),1) + ((1,0),1) &= ((0,0),1), \\
((0,0),1) + ((1,0),1) &= ((1,0),0), \\
((1,0),0) + ((1,0),1) &= ((0,0),0),
\end{aligned}$$

i.e., the element $((1,0),1)$ has order 4.

For a field of characteristic 2 we have the following partial result where the second left F-brace as vector space over F has dimension 1 and the bilinear form comes from a quadratic one.

Corollary 2.4.4. *Let F be a field of characteristic 2, and H a left F-brace, N a left F-brace with additive group $(F, +)$. Let $\mathfrak{q}$ be a quadratic form on H, $\mathfrak{b}$ the polar form of $\mathfrak{q}$, $\beta : N \to \mathrm{Aut}(H)$ a homomorphism of the multiplicative group of N, into the group of automorphisms of the left F-brace H that satisfy the condition*

$$\mathfrak{b}(h_1 \circ {}^n h_2, h_1) = n \circ \mathfrak{q}(h_2) + n + \mathfrak{q}(h_1) + \mathfrak{q}(h_1 \circ {}^n h_2) \tag{2.6}$$

for all $h_1, h_2 \in H$ and $n \in N$. Then the asymmetric product of H by N is a left F-brace with the scalar multiplication given by

$$\mu(h, n) = (\mu h, \mu(\mu + 1)\mathfrak{q}(h) + \mu n), \tag{2.7}$$

for all $h \in H$, $n \in N$ and $\mu \in F$.

Proof. First, (2.6) implies (2.3) in Corollary 2.3.2. In fact, if $h_1, h_2, h_3 \in H$ and $n \in N$, then

$$\begin{aligned}
&\mathfrak{b}\left(h_1 \circ {}^{n} h_2, h_1 \circ {}^{n} h_3\right) + n = \mathfrak{q}\left(h_1 \circ {}^{n} h_2 + h_1 \circ {}^{n} h_3\right) + \mathfrak{q}\left(h_1 \circ {}^{n} h_2\right) + \mathfrak{q}\left(h_1 \circ {}^{n} h_3\right) + n \\
&= \mathfrak{q}\left(h_1 \circ {}^{n} h_2 + h_1 \circ {}^{n} h_3\right) + n \circ \mathfrak{q}\left(h_2\right) + n + \mathfrak{q}\left(h_1\right) + \mathfrak{b}\left(h_1 \circ {}^{n} h_2, h_1\right) \\
&+ n \circ \mathfrak{q}\left(h_3\right) + n + \mathfrak{q}\left(h_1\right) + \mathfrak{b}\left(h_1 \circ {}^{n} h_3, h_1\right) + n \\
&= \mathfrak{q}\left(h_1 \circ {}^{n} h_2 + h_1 \circ {}^{n} h_3\right) + \mathfrak{b}\left(h_1 \circ {}^{n} h_2 + h_1 \circ {}^{n} h_3, h_1\right) + n \circ \left(\mathfrak{q}\left(h_2\right) + \mathfrak{q}\left(h_3\right)\right) \\
&= \mathfrak{q}\left(h_1 \circ {}^{n} h_2 + h_1 \circ {}^{n} h_3 + h_1\right) + \mathfrak{q}\left(h_1\right) + n \circ \left(\mathfrak{b}\left(h_2, h_3\right) + \mathfrak{q}\left(h_2 + h_3\right)\right) \\
&= \mathfrak{q}\left(h_1 \circ {}^{n} \left(h_2 + h_3\right)\right) + \mathfrak{q}\left(h_1\right) + n \circ \mathfrak{b}\left(h_2, h_3\right) + n + n \circ \mathfrak{q}\left(h_2 + h_3\right) \\
&= n \circ \mathfrak{b}\left(h_2, h_3\right) + \mathfrak{b}\left(h_1 \circ {}^{n} \left(h_2 + h_3\right), h_1\right).
\end{aligned}$$

Hence $H \rtimes_{\circ} N$ is a left brace. Moreover $(H \times N, +)$ with the scalar multiplication (2.7) is a vector space over F.

Finally, if $\mu \in F$, $h_1, h_2 \in H$ and $n_1, n_2 \in N$, then

$$\begin{aligned}
&\left(h_1, n_1\right) \circ \left(\mu\left(h_2, n_2\right)\right) + \left(\mu + 1\right)\left(h_1, n_1\right) \\
&= \left(h_1, n_1\right) \circ \left(\mu h_2, \mu\left(\mu + 1\right) \mathfrak{q}\left(h_2\right) + \mu n_2\right) \\
&+ \left(\left(\mu + 1\right) h_1, \left(\mu + 1\right) \mu \mathfrak{q}\left(h_1\right) + \left(\mu + 1\right) n_1\right) \\
&= \left(h_1 \circ {}^{n_1} \left(\mu h_2\right), n_1 \circ \left(\mu\left(\mu + 1\right) \mathfrak{q}\left(h_2\right) + \mu n_2\right)\right) \\
&+ \left(\left(\mu + 1\right) h_1, \left(\mu + 1\right) \mu \mathfrak{q}\left(h_1\right) + \left(\mu + 1\right) n_1\right) \\
&= \left(h_1 \circ {}^{n_1} \left(\mu h_2\right) + \left(\mu + 1\right) h_1, \mathfrak{b}\left(h_1 \circ {}^{n_1} \left(\mu h_2\right), \left(\mu + 1\right) h_1\right)\right. \\
&\left. + n_1 \circ \left(\mu\left(\mu + 1\right) \mathfrak{q}\left(h_2\right) + \mu n_2\right) + \left(\mu + 1\right) \mu \mathfrak{q}\left(h_1\right) + \left(\mu + 1\right) n_1\right) \\
&= \left(\mu\left(h_1 \circ {}^{n_1} h_2\right), \mu\left(\mu + 1\right) \mathfrak{b}\left(h_1 \circ {}^{n_1} h_2, h_1\right) + \mu\left(\mu + 1\right)\left(n_1 \circ \mathfrak{q}\left(h_2\right)\right)\right. \\
&\left. + \left(\mu\left(\mu + 1\right) + 1\right) n_1 + n_1 + n_1 \circ \left(\mu n_2\right) + \mu\left(\mu + 1\right) \mathfrak{q}\left(h_1\right) + \left(\mu + 1\right) n_1\right) \\
&= \left(\mu\left(h_1 \circ {}^{n_1} h_2\right), \mu\left(\mu + 1\right)\left(\mathfrak{b}\left(h_1 \circ {}^{n_1} h_2, h_1\right) + n_1 \circ \mathfrak{q}\left(h_2\right) + n_1 + \mathfrak{q}\left(h_1\right)\right)\right. \\
&\left. + \mu\left(n_1 \circ n_2\right)\right) = \left(\mu\left(h_1 \circ {}^{n_1} h_2\right), \mu\left(\mu + 1\right) \mathfrak{q}\left(h_1 \circ {}^{n_1} h_2\right) + \mu\left(n_1 \circ n_2\right)\right) \\
&= \mu\left(h_1 \circ {}^{n_1} h_2, n_1 \circ n_2\right) = \mu\left(\left(h_1, n_1\right) \circ \left(h_2, n_2\right)\right).
\end{aligned}$$

Therefore $(H \times N, +, \circ)$ is a left F-brace. □

3 The matched product

3.1 The matched product of left semi-braces

We introduce the matched product of left semi-braces, a construction of left semi-braces (to appear in [12]) that generalizes the matched product of left braces introduced in [2] and [4], and includes also the semidirect product of left semi-braces (Corollary 2.1.5).

In this chapter, if B_1, B_2 are groups and $\delta : B_1 \to \mathrm{Sym}(B_2)$ a right action of the group $(B_1, \circ)$ on the set B_2 and $\sigma : B_2 \to \mathrm{Aut}(B_1)$ a left action of the group $(B_2, \circ)$ on the set B_1, then we denote by

$$x^a := \delta_a(x) \qquad \text{and} \qquad {}^x a := \sigma_x(a),$$

for all $a \in B_1$ and $x \in B_2$.

Theorem 3.1.1. *Let B_1, B_2 be left semi-braces, $\delta : B_1 \to \mathrm{Sym}(B_2)$ a right action of the group $(B_1, \circ)$ on the set B_2 and $\sigma : B_2 \to \mathrm{Aut}(B_1)$ a left action of the group $(B_2, \circ)$ on the set B_1 and σ_x is an automorphism of the semigroup $(B_1, +)$, for every $x \in B_2$, such that*

1. ${}^x(a \circ b) = ({}^x a) \circ ({}^{x^a} b)$;
2. ${}^x 0 = 0$;
3. $(x \circ y)^a = x^{{}^y a} \circ y^a$;
4. $0^a = 0$;
5. $\left(\left((x+y)^-\right)^a\right)^- = ((x^-)^a)^- + ((y^-)^a)^-$,

hold for all $a, b \in B_1$ and $x, y \in B_2$. Then the sum and the multiplication over the cartesian product $B_1 \times B_2$ given by

$$(a,x) + (b,y) := (a + b,\ x + y)$$
$$(a,x) \circ (b,y) := \left(a \circ {((x^-)^a)}^- b,\ x \circ \left((y^-)^{\left({}^{x^-} a\right)}\right)^-\right)$$

define a structure of left semi-brace. We call this structure the matched product of the left semi-brace B_1 and the left semi-brace B_2 by δ and σ *and we denote it by* $B_1 \bowtie B_2$.

Proof. Clearly the structure $(B_1 \times B_2, +)$ is a left cancellative semigroup. To prove that $(B_1 \times B_2, \circ)$ is a group, first note that the structure $(B_1 \times B_2, \cdot)$ where

$$(a, x) \cdot (b, y) = \left(a \circ {}^{x}b,\ x^{b} \circ y\right),$$

is a group with identity $(0, 0)$, since it is the matched product of the groups $(B_1, \circ)$ and $(B_2, \circ)$ obtained by the actions σ and δ. In fact, if $a, b, c \in B_1$ and $x, y, z \in B_2$, then, by conditions 1. and 3., we have that

$$\begin{aligned}
&(a, x) \cdot ((b, y) \cdot (c, z)) = (a, x) \cdot (b \circ {}^{y}c, y^{c} \circ z) = \left(a \circ {}^{x}(b \circ {}^{y}c), x^{b \circ {}^{y}c} \circ (y^{c} \circ z)\right) \\
&= \left(a \circ {}^{x}b \circ {}^{x^{b}}({}^{y}c), \left(x^{b}\right)^{{}^{y}c} \circ y^{c} \circ z\right) = \left((a \circ {}^{x}b) \circ {}^{x^{b} \circ {}^{y}c}, \left(x^{b} \circ y\right)^{c} \circ z\right) \\
&= \left(a \circ {}^{x}b, x^{b} \circ y\right) \cdot (c, z) = ((a, x) \cdot (b, y)) \cdot (c, z).
\end{aligned}$$

By condition 2., we have

$$(a, x) \cdot (0, 0) = \left(a \circ {}^{x}0, x^{0} \circ 0\right) = (a \circ 0, x \circ 0) = (a, x),$$

by condition 4.,

$$(0, 0) \cdot (a, x) = \left(0 \circ {}^{0}a, 0^{a} \circ x\right) = (0 \circ a, 0 \circ x) = (a, x)$$

and, since δ and σ are actions,

$$\begin{aligned}
(a, x) \cdot \left({}^{x^{-}}(a^{-}), \left(x^{{}^{x^{-}}(a^{-})}\right)^{-}\right) &= \left(a \circ {}^{x}\left({}^{x^{-}}(a^{-})\right), x^{{}^{x^{-}}(a^{-})} \circ \left(x^{{}^{x^{-}}(a^{-})}\right)^{-}\right) \\
&= (a \circ a^{-}, 0) = (0, 0).
\end{aligned}$$

Moreover, the map $\varphi : B_1 \times B_2 \to B_1 \times B_2$ given by

$$\varphi(a, x) := \left(a,\ \left((x^{-})^{a^{-}}\right)^{-}\right),$$

for every $(a, x) \in B_1 \times B_2$ is an isomorphism from the group $(B_1 \times B_2, \cdot)$ into $(B_1 \times B_2, \circ)$.
First note that, if $c \in B_1$ and $z \in B_2$, then $0 = {}^{z}(c^{-} \circ c) = {}^{z}c^{-} \circ {}^{z^{c^{-}}}c$, that is, $({}^{z}c^{-})^{-} = {}^{z^{c^{-}}}c$ and in particular

$${}^{(z^{c})^{-}}(z c)^{-} = {}^{(z^{c})^{-}}\left({}^{z^{c}}c^{-}\right) = {}^{(z^{c})^{-} \circ (z^{c})}(c^{-}) = c^{-}.$$

Similarly, if $c \in B_1$ and $z \in B_2$, it follows that $0 = (z \circ z^-)^c = {z^{-}}^c \circ (z^-)^c$, i.e., $((z^-)^c)^- = {z^{-}}^c$ in particular

$$\left((z^c)^-\right)^{({}^z c)^-} = \left((z^-)^{{}^z c}\right)^{({}^z c)^-} = z^-.$$

If $a, b \in B_1$ and $x, y \in B_2$, then

$$\begin{aligned}
\varphi(a,x) \circ \varphi(b,y) &= \left(a, \left((x^-)^{a^-}\right)^-\right) \circ \left(b, \left((y^-)^{b^-}\right)^-\right) \\
&= \left(a \circ \left(\left((x^-)^{a^-}\right)^a\right)^- b, \left((x^-)^{a^-}\right)^- \circ \left(\left((y^-)^{b^-}\right)^{\left(\left((x^-)^{a^-}\right)_a\right)^-}\right)^-\right) \\
&= \left(a \circ {}^x b, \left((x^-)^{a^-}\right)^- \circ \left(\left((y^-)^{b^-}\right)^{{}^{x^-}a^-}\right)^-\right)
\end{aligned}$$

and

$$\begin{aligned}
\varphi((a,x)\cdot(b,y)) &= \varphi\left(a \circ {}^x b, x^b \circ y\right) = \left(a \circ {}^x b, \left(\left(x^b \circ y\right)^-\right)^{(a \circ {}^x b)^-}\right)^- \\
&= \left(a \circ {}^x b, \left(\left(y^- \circ \left(x^b\right)^-\right)^{(a \circ {}^x b)^-}\right)^-\right) \\
&= \left(a \circ {}^x b, \left((y^-)^{\left(x^b\right)^- (a \circ {}^x b)^-} \circ \left(\left(x^b\right)^-\right)^{(a \circ {}^x b)^-}\right)^-\right) \\
&= \left(a \circ {}^x b, \left((y^-)^{\left(x^b\right)^- ({}^x b)^- \circ \left(\left(x^b\right)^-\right)^{({}^x b)^-}\, a^-} \circ \left((x^-)^{{}^x b}\right)^{({}^x b)^- \circ a^-}\right)^-\right) \\
&= \left(a \circ {}^x b, \left((y^-)^{b^- \circ {}^{x^-}a^-} \circ (x^-)^{a^-}\right)^-\right) \\
&= \left(a \circ {}^x b, \left((x^-)^{a^-}\right)^- \circ \left(\left((y^-)^{b^-}\right)^{{}^{x^-}a^-}\right)^-\right).
\end{aligned}$$

Hence φ is a homomorphism from the group $(B_1 \times B_2, \cdot)$ into $(B_1 \times B_2, \circ)$. Further, $\psi : B_1 \times B_2 \to B_1 \times B_2$, given by $\psi(a,x) := \left(a, \left({}^{x^-}a\right)^-\right)$, for all $a \in B_1$ and $x \in B_2$, is the inverse of φ, so φ is bijective.

Finally, note that

$$(a,x)^- = \left(\left({}^{x^-}a\right)^-, (x^-)^a\right),$$

for every $(a,x) \in B_1 \times B_2$. If $a,b,c \in B_1$ and $x,y,z \in B_2$, then, by 1., 2., 4. and since σ is a left action f the group $(B_2,\circ)$ on the left semi-brace B_1, we have that

$$
\begin{aligned}
&(a,x)\circ(b,y)+(a,x)\circ\left((a,x)^{-}+(c,z)\right)\\
&=\left(a\circ((x^{-})^{a})^{-}b, x\circ\left((y^{-})^{\left(x^{-}a\right)^{-}}\right)^{-}\right)+(a,x)\circ\left(\left(x^{-}a\right)^{-}+c,(x^{-})^{a}+z\right)\\
&=\left(a\circ((x^{-})^{a})^{-}b, x\circ\left((y^{-})^{\left(x^{-}a\right)^{-}}\right)^{-}\right)\\
&+\left(a\circ((x^{-})^{a})^{-}\left(\left(x^{-}a\right)^{-}+c\right), x\circ\left(\left(\left((x^{-})^{a}+z\right)^{-}\right)^{\left(x^{-}a\right)^{-}}\right)^{-}\right)\\
&=\Bigg(a\circ((x^{-})^{a})^{-}b+a\circ\left(((x^{-})^{a})^{-}\left((x^{-})^{a}a^{-}\right)+((x^{-})^{a})^{-}c\right),\\
&x\circ\left((y^{-})^{\left(x^{-}a\right)^{-}}\right)^{-}+x\circ\left(\left(\left((x^{-})^{a}\right)^{-\left(x^{-}a\right)^{-}}\right)^{-}+\left((z^{-})^{\left(x^{-}a\right)^{-}}\right)^{-}\right)\Bigg)\\
&=\Bigg(a\circ((x^{-})^{a})^{-}b+a\circ\left(a^{-}+((x^{-})^{a})^{-}c\right),\\
&x\circ\left((y^{-})^{\left(x^{-}a\right)^{-}}\right)^{-}+x\circ\left(x^{-}+\left((z^{-})^{\left(x^{-}a\right)^{-}}\right)^{-}\right)\Bigg)\\
&=\left(a\circ((x^{-})^{a})^{-}(b+c), x\circ\left(\left((y^{-})^{\left(x^{-}a\right)^{-}}\right)^{-}+\left((z^{-})^{\left(x^{-}a\right)^{-}}\right)^{-}\right)\right)\\
&=\left(a\circ((x^{-})^{a})^{-}(b+c), x\circ\left((y+z)^{-\left(x^{-}a\right)^{-}}\right)^{-}\right)=(a,x)\circ((b,y)+(c,z)).
\end{aligned}
$$

Therefore the property (1.1) in Definition 1.1.1 is satisfied and so $B_1 \bowtie B_2$ is a left semi-brace. □

Remark 3.1.2. Let B be the matched product of left semi-braces B_1 and B_2. If E_1 and E_2 are the set of idempotents of $(B_1,+)$ and $(B_2,+)$ respectively, then the set E of idempotents of $(B_1 \bowtie B_2,+)$ is $E_1 \times E_2$. In fact, if $e \in E_1$ and $f \in E_2$, then $(e,f)+(e,f)=(e+e,f+f)=(e,f)$, i.e., $(e,f) \in E$. Conversely, if $(a,x) \in E$, then $(a,x)=(a,x)+(a,x)=(a+a,x+x)$, i.e., $a=a+a$ and $x+x=x$. Moreover, since $(0,0)$ is the identity of $(B_1 \bowtie B_2,\circ)$, the group $G := B+(0,0)$ is equal to the direct sum of the groups $G_1 := B_1+0$ and $G_2 := B_2+0$.

Remark 3.1.3. Let B_1 and B_2 be left semi-braces. If $\beta : B_2 \to \mathrm{Aut}(B_1)$ is a left action of the group $(B_2,\circ)$ on the left semi-brace B_1, then, by the Corollary 2.1.5 we have that $B_1 \rtimes B_2$ is a left semi-brace. Note that this construction can be see

as a corollary of the previous theorem. In fact, set $\sigma := \beta$, and $\delta : B_1 \to \mathrm{Sym}(B_2)$ defined by $\delta(a) = \mathrm{id}_{B_2}$, for every $a \in B_1$. Then conditions 1., 2., 3. and 4. are satisfied. Moreover the sum and the product over the cartisian product are

$$(a,x)+(b,y) = (a+b, x+y)$$
$$(a,x)\circ(b,y) = (a\circ {}^{x}b, x\circ y)$$

for all $a,b \in B_2$ and $x,y \in B_2$. Exactly the same obtained in Corollary 2.1.5.

Lemma 3.1.4. *Let B be a left semi-brace, E the set of idempotents of $(B,+)$, 0 the identity of $(B,\circ)$ and $G := B+0$. The following statements hold:*

1. *if H is a subgroup of $(G,+)$ such that $\rho_b(H) \subseteq H$, for every $b \in B$, then $(H,+,\circ)$ is a skew left brace;*

2. *if N is a subgroup of $(G,+)$ such that $\lambda_b(N) \subseteq N+E$, for every $b \in B$, then $(N+E,+,\circ)$ is a left semi-brace.*

Proof. 1. First note that H is not empty. If $h \in H$, then

$$\begin{aligned} h^- &= \left(0+h+(-h)\circ(-h)^-\right)^- = \left((-h)\circ\left((-h)^- + (-h)^-\right)\right)^- \\ &= \left((-h)^- + (-h)^-\right)\circ(-h)^- = \rho_{(-h)^-}(-h) \in H. \end{aligned}$$

Moreover if $h_1, h_2 \in H$, then

$$\begin{aligned} \left(\rho_{h_1^-}\left(h_2^-\right)\right)^- + h_1 &= \left(\left(h_2+h_1^-\right)^-\circ h_1^-\right)^- + h_1 = h_1\circ\left(h_2+h_1^-\right)+h_1 \\ &= h_1\circ h_2 - h_1 + h_1\circ h_1^- + h_1 = h_1\circ h_2 + 0 = h_1\circ h_2, \end{aligned}$$

i.e., $h_1\circ h_2 \in H$. Hence $(H,\circ)$ is a subgroup of $(B,\circ)$. Finally the property (1.1) in Definition 1.1.1 holds. Therefore $(H,+,\circ)$ is a skew left brace, since, by assumption, $(H,+)$ is a group.
2. Clearly $(N+E,+)$ is a right group, in particular is a left cancellative semigroup. First if $n \in N$, then $n^- = -\lambda_{n^-}(n) \in (N+E)\cap G = N$ and so if $e \in E$ we have that

$$\begin{aligned} (n+e)^- &= (n\circ\lambda_{n^-}(e))^- = (\lambda_{n^-}(e))^-\circ n^- = (\lambda_{n^-}(e))^- + \lambda_{(\lambda_{n^-}(e))^-}(n^-) \\ &= \lambda_{(\lambda_{n^-}(e))^-}(n^-) \in N+E. \end{aligned}$$

Further, if $n_1, n_2 \in N$ and $e_1, e_2 \in N$, then

$$\begin{aligned} (n_1+e_1)\circ(n_2+e_2) &= (n_1+e_1)+\lambda_{n_1+e_1}(n_2+e_2) \\ &= n_1+\lambda_{n_1+e_1}(n_2)+\lambda_{n_1+e_1}(e_2) \in N+E. \end{aligned}$$

Hence $(N+E,\circ)$ is a subgroup of $(B,\circ)$, moreover $(N+E,+,\circ)$ satisfies (1.1) in Definition 1.1.1. Therefore $(N+E,+,\circ)$ is a left semi-brace. □

Theorem 3.1.5. *Let B be a left semi-brace, E the set of idempotents of $(B,+)$, 0 the identity of $(B,\circ)$, $G := B+0$, H and N normal subgroups of $(G,+)$ such that $\rho_b(H) \subseteq H$ and $\lambda_b(N) \subseteq N+E$, for every $b \in B$. If $H \cap N = \{0\}$ and $H+N = G$, then B is isomorphic to the matched product of the left semi-braces H and $N+E$.*

Proof. By Lemma 3.1.4, H is a skew left brace and $N+E$ is a left semi-brace. First note that

$$\left(\lambda_{h^-}\left((n+e)^-\right)\right)^- = \left(\lambda_{h^-}\left((n+e)^- \circ e\right) + \lambda_{h^-}\left((n+e)^- \circ n\right)\right)^- \in N+E,$$

by Lemma 2.1.1.1 and

$$\left(\rho_{(n+e)^-}\left(h^-\right)\right) \in H,$$

for all $h \in H$, $n \in N$, $e \in E$. Let $\delta_h : N+E \to N+E$ the map defined by

$$\delta_h(n+e) := \left(\lambda_{h^-}\left((n+e)^-\right)\right)^-,$$

for every $h \in H$ and $\sigma_{n+e} : H \to H$ the map defined by

$$\sigma_{n+e}(h) := \left(\rho_{(n+e)^-}\left(h^-\right)\right)^-,$$

for every $n+e \in N+E$. Moreover δ_h and σ_{n+e} are bijective. In fact,

$$\delta_0(n+e) = \left(\lambda_0\left((n+e)^-\right)\right)^- = \left(0 \circ \left(0 + (n+e)^-\right)\right)^- = n+e,$$

for every $n+e \in N+E$, i.e., $\delta_0 = \mathrm{id}_{N+E}$ and if $h_1, h_2 \in H$ then

$$\begin{aligned}\delta_{h_2}\left(\delta_{h_1}(n+e)\right) &= \left(\lambda_{h_2^-}\left(\lambda_{h_1^-}(n+e)\right)\right)^- = \left(\lambda_{h_2^- \circ h_1^-}\left((n+e)^-\right)\right)^- \\ &= \left(\lambda_{(h_1 \circ h_2)^-}\left((n+e)^-\right)\right)^- = \delta_{h_1 \circ h_2}(n+e),\end{aligned}$$

by Proposition 1.2.1, for every $n+e \in N+E$, hence δ_h is bijective, for every $h \in H$. Further, if $h \in H$

$$\sigma_0(h) = \left(\rho_0\left(h^-\right)\right)^- = \left((h+0)^- \circ 0\right)^- = h,$$

since $h \in G$ and

$$\begin{aligned}\sigma_{n_1+e_1}\left(\sigma_{n_2+e_2}(h)\right) &= \left(\rho_{(n_1+e_1)^-}\left(\rho_{(n_2+e_2)^-}\left(h^-\right)\right)\right)^- = \left(\rho_{(n_2+e_2)^- \circ (n_1+e_1)^-}\left(h^-\right)\right)^- \\ &= \left(\rho_{((n_1+e_1)\circ(n_2+e_2))^-}\left(h^-\right)\right)^- = \sigma_{(n_1+e_1)\circ(n_2+e_2)}(h),\end{aligned}$$

by Proposition 1.2.9, for all $n_1+e_1, n_2+e_2 \in N+E$. Hence σ_{n+e} is bijective for every $n+e \in N+E$. Moreover if $n+e \in N+E$ and $h_1, h_2 \in H$ then

$$\begin{aligned}
&\sigma_{n+e}(h_1)+\sigma_{n+e}(h_2) = \left(\rho_{(n+e)^-}\left(h_1^-\right)\right)^- + \left(\rho_{(n+e)^-}\left(h_2^-\right)\right)^- \\
&= (n+e)\circ\left(h_1+(n+e)^-\right)+(n+e)\circ\left(h_2+(n+e)^-\right) \\
&= (n+e)\circ h_1+(n+e)\circ\left((n+e)^-+(n+e)^-\right) \\
&+(n+e)+(n+e)\circ\left((n+e)^-+h_2+(n+e)^-\right) \\
&= (n+e)\circ h_1+(n+e)\circ\left((n+e)^-+h_2+(n+e)^-\right) \\
&= (n+e)\circ\left(h_1+h_2+(n+e)^-\right) = \left(\rho_{(n+e)^-}\left((h_1+h_2)^-\right)\right)^- \\
&= \sigma_{n+e}(h_1+h_2),
\end{aligned}$$

since $(n+e)\circ\left((n+e)^-+(n+e)^-\right)+(n+e) \in E$ in fact

$$\begin{aligned}
&\left((n+e)\circ\left((n+e)^-+(n+e)^-\right)+(n+e)\right)+\left((n+e)\circ\left((n+e)^-+(n+e)^-\right)\right. \\
&\left.+(n+e)\right) = (n+e)\circ\left((n+e)^-+(n+e)^-\right)+(n+e)\circ\left(0+(n+e)^-\right)+(n+e) \\
&= (n+e)\circ\left((n+e)^-+(n+e)^-\right)+0+(n+e) \\
&= (n+e)\circ\left((n+e)^-+(n+e)^-\right)+(n+e),
\end{aligned}$$

i.e., $\sigma_{n+e} \in \mathrm{Aut}(H)$ is an automorphism of the group $(H,+)$. Therefore we may define a right action $\delta : H \to \mathrm{Sym}(N+E)$ of the group $(H,\circ)$ on the set $N+E$ defined by $(n+e)^h := \delta_h(n+e)$, for every $n+e \in N+E$ and a left action $\sigma : N+E \to \mathrm{Aut}(H)$ of the group $(N+E,\circ)$ on the se H with σ_{n+e} an automorphism of the group $(H,+)$, defined by ${}^{(n+e)}h := \sigma_{n+e}(h)$. Further if $n+e \in N+E$, $h_1, h_2 \in H$, then

$$\begin{aligned}
&\left({}^{(n+e)}h_1\right)\circ\left({}^{(n+e)^{h_1}}h_2\right) = \left(\rho_{(n+e)^-}\left(h_1^-\right)\right)^- \circ \left(\rho_{\lambda_{h_1^-}\left((n+e)^-\right)}\left(h_2^-\right)\right)^- \\
&= (n+e)\circ\left(h_1+(n+e)^-\right)\circ\left(h_1+(n+e)^-\right)^- \circ h_1 \circ \left(h_2+h_1^-\circ\left(h_1+(n+e)^-\right)\right) \\
&= (n+e)\circ h_1\circ h_1^-\circ\left(h_1\circ h_2+(n+e)^-\right) = \left(\left(h_1\circ h_2+(n+e)^-\right)^-\circ(n+e)^-\right)^- \\
&= \left(\rho_{(n+e)^-}\left((h_1\circ h_2)^-\right)\right) = {}^{(n+e)}(h_1\circ h_2),
\end{aligned}$$

i.e., condition 1. in Theorem 3.1.1 holds. For 2., if $n+e \in N+E$, then

$${}^{(n+e)}0 = \left(\rho_{(n+e)^-}(0)\right)^- = \left(\left(0+(n+e)^-\right)^-\circ(n+e)^-\right)^- = 0.$$

Moreover if $n_1+e_1, n_2+e_2 \in N+E$ and $h \in H$, then

$$\begin{aligned}
&(n_1+e_1)^{(n_2+e_2)h} \circ (n_2+e_2)^h \\
&= \left(\lambda_{\rho_{(n_2+e_2)^-}(h^-)}\left((n_1+e_1)^-\right)\right)^- \circ \left(\lambda_{h^-}\left((n_2+e_2)^-\right)\right)^- \\
&= \left(h^- \circ \left(h+(n_2+e_2)^-\right) \circ \left(h+(n_2+e_2)^-\right)^- \circ (n_2\right. \\
&\left.+e_2)^- \circ \left((n_2+e_2) \circ \left(h+(n_2+e_2)^-\right)+(n_1+e_1)^-\right)\right)^- \\
&= \left(h^- \circ (n_2+e_2)^- \circ \left((n_2+e_2) \circ \left(h+(n_2+e_2)^-\right)+(n_1+e_1)^-\right)\right)^- \\
&= \left(h^- \circ \left(h+(n_2+e_2)^- + (n_2+e_2)^- \circ \left((n_2+e_2)+(n_1+e_1)^-\right)\right)\right)^- \\
&= \left(h^- \circ \left(h+(n_2+e_2)^- \circ (n_1+e_1)^-\right)\right) \\
&= \left(\lambda_{h^-}\left(((n_1+e_1) \circ (n_2+e_2))^-\right)\right)^- = ((n_1+e_1) \circ (n_2+e_2))^h,
\end{aligned}$$

i.e., condition 3. in Theorem 3.1.1 holds. Furthermore for condition 4. if $h \in H$, then

$$0^h = \left(\lambda_{h^-}(0)\right)^- = \left(h^- \circ (h+0)\right)^- = 0.$$

Finally if $n_1+e_1, n_2+e_2 \in N+E$ and $h \in H$, then

$$\begin{aligned}
\left(\left(((n_1+e_1)+(n_2+e_2))^-\right)^h\right)^- &= \lambda_{h^-}(n_1+e_1+n_2+e_2) \\
&= \lambda_{h^-}(n_1+e_1) + \lambda_{h^-}(n_2+e_2) \\
&= \left(\left((n_1+e_1)^-\right)^h\right)^- + \left(\left((n_2+e_2)^-\right)^h\right)^-,
\end{aligned}$$

i.e., condition 5. in Theorem 3.1.1 holds. Finally, $\varphi : H \times (N+E) \to B$ defined by

$$\varphi(h, n+e) := h+n+e$$

is an isomorphism from the left semi-brace $H \bowtie (N+E)$ into the left semi-brace B. In fact, if $(h_1, n_1+e_1), (h_2, n_2+e_2) \in H \times (N+E)$, then

$$\begin{aligned}
&\varphi(h_1, n_1+e_1) + \varphi(h_2, n_2+e_2) = h_1+n_1+e_1+h_2+n_2+e_2 \\
&= h_1+n_1+h_2+n_2+e_2 = h_1+h_2+n_1+n_2+e_2 = \varphi(h_1+h_2, n_1+n_2+e_2) \\
&= \varphi((h_1, n_1+e_1)+(h_2, n_2+e_2)),
\end{aligned}$$

since, by assumptions, $n_2+h_2 = h_2+n_2$, moreover

$$
\begin{aligned}
&\varphi\left((h_1, n_1+e_1)\circ(h_2, n_2+e_2)\right)\\
&=\varphi\left(h_1\circ{}^{\left(\left((n_1+e_1)^-\right)^{h_1}\right)^-}h_2, (n_1+e_1)\circ\left(\left((n_2+e_2)^-\right)^{\left((n_1+e_1)^-h_1\right)^-}\right)^-\right)\\
&=\varphi\left(h_1\circ\left(\rho_{\left(\lambda_{h_1^-}(n_1+e_1)\right)^-}\left(h_2^-\right)\right)^-, (n_1+e_1)\circ\left(\lambda_{\left(\rho_{n_1+e_1}(h_1^-)\right)^-}(n_2+e_2)\right)\right)\\
&=\varphi\Big(h_1\circ h_1^-\circ(h_1+n_1+e_1)\circ\left(h_2+(h_1+n_1+e_1)^-\circ h_1\right),\\
&(n_1+e_1)\circ(n_1+e_1)^-\circ(h_1+n_1+e_1)\circ\left((h_1+n_1+e_1)^-\circ(n_1+e_1)+n_2+e_2\right)\Big)\\
&=(h_1+n_1+e_1)\circ\left(h_2+(h_1+n_1+e_1)^-\circ h_1\right)+(h_1+n_1+e_1)\circ((h_1+n_1\\
&+e_1)^-+(h_1+n_1+e_1)^-\circ((h_1+n_1+e_1)+(n_1+e_1))+n_2+e_2\Big)\\
&=(h_1+n_1+e_1)\circ\Big(h_2+(h_1+n_1+e_1)^-\circ h_1+(h_1+n_1+e_1)^-\circ((h_1+n_1+e_1)\\
&+n_1+e_1)+n_2+e_2)\\
&=(h_1+n_1+e_1)\circ\left(h_2+(h_1+n_1+e_1)^-\circ(h_1+n_1+e_1)+n_2+e_2\right)\\
&=(h_1+n_1+e_1)\circ(h_2+n_2+e_2)=\varphi(h_1, n_1+e_1)\circ\varphi(h_2, n_2+e_2).
\end{aligned}
$$

Hence φ is a homomorphism from $H\bowtie(N+E)$ into B. Further if $h_1, h_2\in H$, $n_1, n_2\in N$ and $e_1, e_2\in E$ such that $\varphi(h_1, n_1+e_1)=\varphi(h_2, n_2+e_2)$, then $h_1+n_1+e_1=h_2+n_2+e_2$. It follows that $h_1+n_1=h_2+n_2$, i.e., $h_1-h_2=n_2-n_1$ and so $h_1=h_2$ and $n_1=n_2$, since $H\cap N=\{0\}$, moreover $e_1=e_2$. Hence φ is injective. If $b\in B$, then, by Proposition 1.2.4, there exist $g\in G$ and $e\in E$ such that $b=g+e$, further there exist $h\in H$ and $n\in N$, such that $g=h+n$, since $G=H+N$. Hence

$$\varphi(h, n+e)=h+n+e=g+e=b,$$

i.e., φ is surjective. Therefore B is isomorphic to $H\bowtie(N+E)$. □

3.2 The matched product of skew left braces

Corollary 3.2.1. *Let B_1, B_2 be skew left braces, $\delta: B_1\to\mathrm{Sym}(B_2)$ a right action of the group $(B_1,\circ)$ on the set B_2 and $\sigma: B_2\to\mathrm{Aut}(B_1)$ a left action of the group $(B_2,\circ)$ on the set B_1 and σ_h is an automorphism of the group $(B_1,+)$, for every $h\in B_2$, such that*

1. ${}^h(g_1\circ g_2)=\left({}^hg_1\right)\circ\left({}^{h^{g_1}}g_2\right)$;

2. $(h_1\circ h_2)^g={h_1}^{{}^{h_2}g}\circ {h_2}^g$;

3. $\left(\left((h_1+h_2)^-\right)^g\right)^- = \left((h_1{}^-)^g\right)^- + \left((h_2{}^-)^g\right)^-$,

hold for all $g, g_1, g_2 \in B_1$ *and* $h, h_1, h_2 \in B_2$. *Then the sum and the multiplication over the cartesian product* $B_1 \times B_2$ *given by*

$$(g_1,h_1)+(g_2,h_2) := (g_1+g_2,\ h_1+h_2)$$
$$(g_1,h_1)\circ(g_2,h_2) := \left(g_1 \circ \left((h_1{}^-)^{g_1}\right)^- g_2,\ h_1 \circ \left((h_2{}^-)^{\left(h_1{}^- g_1\right)^-}\right)^-\right)$$

define a structure of skew left brace. We call this structure the matched product of the skew left brace B_1 and the skew left brace B_2 by δ and σ *and we denote it by* $B_1 \bowtie B_2$.

Proof. If $h \in B_2$, then, since σ is an action of the group $(B_2, \circ)$ on the group $(B_2, +)$,

$$^h 0 = {}^h(0+0) = {}^h 0 + {}^h 0,$$

i.e., $^h 0 = 0$, and so the condition 2. in Theorem 3.1.1 is satisfied. Moreover, if $g \in B_1$, then by condition 3.,

$$\begin{aligned} 0^g = (0^-)^g = \left((0+0)^-\right)^g &= \left(\left((0^-)^g\right)^- + \left((0^-)^g\right)^-\right)^- \\ &= \left((0^g)^- + (0^g)^-\right)^-, \end{aligned}$$

that is $(0^g)^- = (0^g)^- + (0^g)^-$ and, since $(B_2, +)$ is a group, $0^g = 0$. Hence the condition 4. in Theorem 3.1.1 is satisfied. Therefore we may consider the matched product $B_1 \bowtie B_2$ of the skew left braces B_1 and B_2 by δ and σ . In particular, the left semi-brace $B_1 \bowtie B_2$ is a skew left brace, by Corollary 1.2.8, since $(B_1 \bowtie B_2, +)$ is a group. □

We may also provide suitable conditions for certain subgroups $(G, +)$ and $(H, +)$ of the additive group of a skew left brace B in order to have B as matched product of G and H as skew left braces.

Corollary 3.2.2. *Let* B *be a skew left brace,* H, N *normal subgroups of* $(B, +)$ *such that* $\lambda_b(H) \subseteq H$ *and* $\lambda_b(N) \subseteq N$, *for every* $b \in B$. *If* $H \cap N = \{0\}$ *and* $H + N = B$, *then* B *is isomorphic to the matched product of the skew left braces* H *and* N.

Proof. First note that $\rho_b(H) \subseteq H$ for every $b \in B$. In fact if $h \in H$ and $b \in B$, then $h^- = -\lambda_{h^-}(h) \in H$ and

$$\begin{aligned}\rho_b(h) = (h^- + b)^- \circ b = (b^- \circ (h^- + b))^- &= (b^- \circ h^- - b^-)^- \\ &= (b^- + \lambda_{b^-}(h^-) - b^-) \in H,\end{aligned}$$

since H is a normal subgroup of $(B, +)$. Hence, by Theorem 3.1.5, B is isomorphic to the matched product of skew left braces H and N. □

3.3 The matched product of left braces

Similarly, we may obtain new left braces from two given left braces.

Corollary 3.3.1. *Let B_1, B_2 be left braces, $\delta : B_1 \to \mathrm{Sym}(B_2)$ a right action of the group $(B_1, \circ)$ on the set B_2 and $\sigma : B_2 \to \mathrm{Aut}(B_1)$ a left action of the group $(B_2, \circ)$ on the set B_1 and σ_h is an automorphism of the group $(B_1, +)$, for every $h \in B_2$, such that*

1. ${}^h(g_1 \circ g_2) = \left({}^h g_1\right) \circ \left({}^{h^{g_1}} g_2\right)$;
2. $(h_1 \circ h_2)^g = {h_1}^{{}^{h_2} g} \circ {h_2}^g$;
3. $\left(\left((h_1 + h_2)^-\right)^g\right)^- = \left((h_1{}^-)^g\right)^- + \left((h_2{}^-)^g\right)^-$,

hold for all $g, g_1, g_2 \in B_1$ and $h, h_1, h_2 \in B_2$. Then the sum and the multiplication over the cartesian product $B_1 \times B_2$ given by

$$\begin{aligned}(g_1, h_1) + (g_2, h_2) &:= (g_1 + g_2,\ h_1 + h_2) \\ (g_1, h_1) \circ (g_2, h_2) &:= \left(g_1 \circ {((h_1{}^-)^{g_1})^-} g_2,\ h_1 \circ \left((h_2{}^-)^{\left({h_1}^- g_1\right)^-}\right)^-\right)\end{aligned}$$

define a structure of left brace. We call this structure the matched product of the left brace B_1 and the left brace B_2 by δ and σ *and we denote it by $B_1 \bowtie B_2$.*

Proof. By Corollary 3.2.1, $B_1 \bowtie B_2$ is a skew left brace, moreover $(B_1 \bowtie B_2, +)$ is an abelian group. Therefore $B_1 \bowtie B_2$ is a left brace. □

Corollary 3.3.2. *Let B be a left brace, H, N subgroups of $(B, +)$ such that $\lambda_b(H) \subseteq H$ and $\lambda_b(N) \subseteq N$, for every $b \in B$. If $H \cap N = \{0\}$ and $H + N = B$, then B is isomorphic to the matched product of the left braces H and N.*

Proof. Since $(B,+)$ is an abelian group, clearly H, N are normal subgroup of $(B,+)$ and they are abelian. Hence, $(H,+,\circ)$ and $(N,+,\circ)$ are left braces and by Corollary 3.2.2, B and $H \bowtie N$ are isomorphic as left braces. □

Remark 3.3.3. Let B be a left brace, H, N subgroups of $(B,+)$ such that $H \cap N = \{0\}$, $H + N = B$, $\lambda_b(H) \subseteq H$ and $\lambda_b(N) \subseteq N$, for every $b \in B$. If $h \in H$ and $n \in N$, then the actions introduced in the proof of Theorem 3.1.5:

$$n^h = \delta_h(n) = \left(\lambda_{h^-}\left(n^-\right)\right)^-$$
$$^n h = \sigma_n(h) = \left(\rho_{n^-}\left(h^-\right)\right)^-$$

become

$$n^h = \left(\lambda_{h^-}\left(n^-\right)\right)^- = \left(h^- \circ \left(h + n^-\right)\right)^- = \left(n^- + h\right)^- \circ h = \rho_h(n)$$
$$^n h = \left(\rho_{n^-}\left(h^-\right)\right) = \left(\left(h + n^-\right)^- \circ n^-\right)^- = n \circ \left(n^- + h\right) = \lambda_n(h).$$

In particular, we obtain the converse in Theorem 4.2 of [2].

The result in Corollary 3.3.1 coincides with that described in Theorem 4.2 in [2]. For this purpose, we recall the definition of *matched pair of left braces*.

Definition 3.3.4 (Definition 4.1 in [2])**.** Let B_1, B_2 be left braces. Let $\alpha : B_2 \to \mathrm{Aut}(B_1)$ be a homomorphism from the group $(B_2,\circ)$ into the group of automorphism of $(B_1,+)$ and $\beta : B_1 \to \mathrm{Aut}(B_2)$ be a homomorphism from the group $(B_1,\circ)$ into the group of automorphism of $(B_2,+)$. We say that $(B_1, B_2, \alpha, \beta)$ is a *matched pair of left braces* if α and β satisfy

$$\lambda_g \alpha_h = \alpha_{\beta_g(h)} \lambda_{\alpha^{-1}_{\beta_g(h)}(g)} \tag{3.1}$$

and

$$\beta_{\alpha_h(g)} \lambda_{\beta^{-1}_{\alpha_h(g)}(h)} = \lambda_h \beta_g \tag{3.2}$$

where $\alpha(h) = \alpha_h$ and $\beta(g) = \beta_g$, for all $g \in B_1$ and $h \in B_2$.

In Remark 4.4 of [2] it is proved that if $(B_1, B_2, \alpha, \beta)$ is a matched pair of left braces then the function $\sigma_h : B_1 \to B_1,\ g \mapsto \alpha_h(g) =: {}^h g$ is an automorphism of the group $(B_1,+)$ and $\delta_g : B_2 \to B_2,\ h \mapsto \beta^{-1}_{\alpha_h(g)}(h) =: h^g$ is a bijection of the set B_2. Moreover, $\sigma : B_2 \to \mathrm{Aut}(B_1), h \mapsto \sigma_h$ is a left action of the group $(B_2,\circ)$ on the set

B_1 and $\delta : B_1 \to \mathrm{Sym}(B_2), g \mapsto \delta_g$ is a right action of the group $(B_1, \circ)$ on the set B_2. Moreover, if $g_1, g_2 \in B_1$ and $h \in B_1$, then, by (3.1),

$$\begin{aligned} &{}^{h}(g_1 \circ g_2) = \alpha_h(g_1 + \lambda_{g_1}(g_2)) = \alpha_h(g_1) + \alpha_h\lambda_{g_1}(g_2) \\ &= \alpha_h(g_1) + \lambda_{\alpha_h(g_1)}\alpha_{\beta^{-1}_{\alpha_h(g_1)}(h)}(g_2) = {}^{h}g_1 + \lambda_{{}^{h}g_1}\alpha_{g_1^h}(g_2) \\ &= {}^{h}g_1 \circ \alpha_{g_1^h}(g_2) = {}^{h}g_1 \circ {}^{g_1^h}g_2, \end{aligned}$$

i.e., condition 1. in Corollary 3.3.1 holds. If $g \in B_1$ and $h_1, h_2 \in B_2$, then

$$\begin{aligned} &(h_1 \circ h_2)^g = \beta^{-1}_{\alpha_{h_1\circ h_2}(g)}(h_1 \circ h_2) = \beta^{-1}_{\alpha_{h_1}\alpha_{h_2}(g)}(h_1 + \lambda_{h_1}(h_2)) \\ &= \beta^{-1}_{\alpha_{h_1}\alpha_{h_2}(g)}(h_1) + \beta^{-1}_{\alpha_{h_1}\alpha_{h_2}(g)}(\lambda_{h_1}(h_2)) \\ &= \beta^{-1}_{\alpha_{h_1}\alpha_{h_2}(g)}(h_1) + \lambda_{\beta^{-1}_{\alpha_{h_1}\alpha_{h_2}(g)}(h_1)}\beta^{-1}_{\alpha_{h_2}(g)}(h_2) = (h_1)^{\alpha_{h_2}(g)} \circ h_2^g \\ &= h_1^{{}^{h_2}g} \circ h_2^g, \end{aligned}$$

i.e., condition 2. in Corollary 3.3.1 holds. In particular, if $g \in B_1$ and $h \in B_2$, then

$$\left(h^-\right)^{{}^{h}g} \circ h^g = \left(h^- \circ h\right)^g = 0^g = 0,$$

i.e., $\left(h^-\right)^{{}^{h}g} = (h^g)^-$ and so $(h^g)^- = \beta^{-1}_{\alpha_{h^-}(\alpha_h(g))}(h^-) = \beta_g^{-1}(h^-) = \beta_{g^-}(h^-)$. Hence if $g \in B_1$ and $h_1, h_2 \in B_2$, then

$$\begin{aligned} \left(\left((h_1 + h_2)^-\right)^g\right)^- &= \beta_{g^-}(h_1 + h_2) = \beta_{g^-}(h_1) + \beta_{g^-}(h_2) \\ &= \left(\left(h_1^-\right)^g\right)^- + \left(\left(h_2^-\right)^g\right)^-, \end{aligned}$$

i.e., condition 3. in Corollary 3.3.1 holds. Therefore we may define the matched product of the left braces B_1 and B_2 by δ and σ.

Conversely, let G and H be left braces and $B := G \bowtie H$ the matched product of G and H by σ and δ. Let $\alpha : H \to \mathrm{Aut}(G)$ defined by $\alpha_h := \sigma_h$, for every $h \in H$. Then, by assumptions, α_h is an automorphism of $(G, +)$, for every $h \in H$. Moreover, since σ is a left action of the group $(H, \circ)$ on the set G,

$$\alpha_{h_1\circ h_2}(g) = {}^{h_1\circ h_2}g = {}^{h_1}\left({}^{h_2}g\right) = \alpha_{h_1}(\alpha_{h_2}(g)),$$

for all $h_1, h_2 \in H$ and $g \in G$, and so α is a group homomorphism from the group $(H, \circ)$ into the group of automorphism of the group $(G, +)$.

If $g \in G$, define $\beta_g(h) := \left((h^-)^{g^-}\right)^-$, for every $h \in H$. Then β_g is an endomorphism of $(H,+)$. In fact, by condition 3. in Corollary 3.3.1,

$$\begin{aligned}\beta_g(h_1+h_2) &= \left(\left((h_1+h_2)^-\right)^{g^-}\right)^- = \left(\left(h_1^-\right)^{g^-}\right)^- + \left(\left(h_2^-\right)^{g^-}\right)^- \\ &= \beta_g(h_1) + \beta_g(h_2),\end{aligned}$$

for all $g \in G$ and $h_1, h_2 \in H$. Moreover

$$\begin{aligned}\beta_{g_1}(\beta_{g_2}(h)) &= \beta_{g_1}\left(\left((h^-)^{g_2^-}\right)^-\right) = \left(\left((h^-)^{g_2^-}\right)^{g_1^-}\right)^- = \left((h^-)^{g_2^- \circ g_1^-}\right)^- \\ &= \left((h^-)^{(g_1 \circ g_2)^-}\right)^- = \beta_{g_1 \circ g_2}(h)\end{aligned}$$

for all $g_1, g_2 \in G$ and $h \in H$ and $\beta_0(h) = \left((h^-)^0\right)^- = h = \mathrm{id}_H(h)$, for every $h \in H$. Hence $\beta : G \to \mathrm{Aut}(H)$ is a homomorphism from the group $(G,\circ)$ into the group of automorphism of $(H,+)$.
Finally we check that (3.1) and (3.2) in Definition 3.3.4 are satisfied. First, note that

$$\begin{aligned}&\alpha_{\beta_g(h)}\left(\alpha^{-1}_{\beta_g(h)}(g) \circ t\right) = \alpha_{\beta_g(h)}\left(\alpha_{(\beta_g(h))^-}(g) \circ t\right) = {\left((h^-)^{g^-}\right)^-}\left(\left((h^-)^{g^-} g\right) \circ t\right) \\ &= g \circ {\left(\left((h^-)^{g^-}\right)^-\right)}^{\left((h^-)^{g^-} g\right)} t = g \circ {}^{h}t = g \circ (\alpha_h(t)),\end{aligned}$$

for all $g, t \in G$ and $h \in H$, since

$$\left(\left((h^-)^{g^-}\right)^-\right)^{\left((h^-)^{g^-} g\right)} = \left(h^{h^- g^-}\right)^{(h^-)^{g^-} g} = h^{h^- g^- \circ (h^-)^{g^-} g} = h^{h^-(g^- \circ g)} = h.$$

Then

$$\begin{aligned}&\alpha_{\beta_g(h)}\lambda_{\alpha^{-1}_{\beta_g(h)}(g)}(t) = \alpha_{\beta_g(h)}\left(-\alpha^{-1}_{\beta_g(h)}(g) + \alpha^{-1}_{\beta_g(h)}(g) \circ t\right) \\ &= -\alpha_{\beta_g(h)}\left(\alpha^{-1}_{\beta_g(h)}(g)\right) + \alpha_{\beta_g(h)}\left(\alpha^{-1}_{\beta_g(h)}(g) \circ t\right) = -g + g \circ (\alpha_h(t)) = \lambda_g \alpha_h(t)\end{aligned}$$

for all $g,t \in G$ and $h \in H$, i.e., condition (3.1) holds. Moreover , if $g \in G$ and $h,u \in H$, then

$$\begin{aligned}
&\beta_{\alpha_h(g)}\left(\beta^{-1}_{\alpha_h(g)}(h)\circ u\right) = \beta_{\alpha_h(g)}\left(\beta_{(\alpha_h(g))^-}(h)\circ u\right)\\
&= \left(\left(\left(\left((h^-)^{{}^hg}\right)^-\circ u\right)^-\right)^{({}^hg)^-}\right)^- = \left(\left(u^-\circ (h^-)^{{}^hg}\right)^{({}^hg)^-}\right)^-\\
&= \left((u^-)^{(h^-)^{{}^hg}\left(({}^hg)^-\right)}\circ h^-\right)^- = h\circ\left((u^-)^{g^-}\right)^- = h\circ(\beta_g(u)),
\end{aligned}$$

since

$$(h^-)^{{}^hg}\left(\left({}^hg\right)^-\right) = (h^-)^{{}^hg}\left({}^{h^g}g^-\right) = (h^-)^{{}^hg\circ h^g}g^- = (h^-\circ h)^g g^- = g^-.$$

Hence

$$\begin{aligned}
&\beta_{\alpha_h(g)}\lambda_{\beta^{-1}_{\alpha_h(g)}(h)}(u) = \beta_{\alpha_h(g)}\left(-\beta^{-1}_{\alpha_h(g)}(h)+\beta^{-1}_{\alpha_h(g)}(h)\circ u\right)\\
&= -h+\beta_{\alpha_h(g)}\left(\beta^{-1}_{\alpha_h(g)}(h)\circ u\right) = -h+h\circ\beta_g(u) = \lambda_h(\beta_g(u)),
\end{aligned}$$

for all $g \in G$ and $h,u \in H$, i.e., condition (3.2) holds. Therefore, (G,H,α,β) is a *matched pair of left braces*. Finally, to see that the multiplications are the same we may compute the lambda functions. We have

$$\begin{aligned}
&\lambda_{(g,h)}(t,u) = -(g,h)+(g,h)\circ(t,u)\\
&= (-g,-h)+\left(g\circ((h^-)^g)^-t,\ \left(u^{-\left((h^-)g\right)^-}\circ h^-\right)^-\right)\\
&= \left(-g+g\circ((h^-)^g)^-t,\ -h+h\circ\left((u^-)^{\left((h^-)g\right)^-}\right)^-\right)\\
&= \left(\lambda_g\left(((h^-)^g)^-t\right),\ \lambda_h\left(\left((u^-)^{\left((h^-)g\right)^-}\right)^-\right)\right)\\
&= \left(\lambda_g\left(\alpha_{((h^-)^g)^-}(t)\right),\ \lambda_h\left(\beta_{(h^-g)}(u)\right)\right)\\
&= \left(\lambda_g\left(\alpha_{\beta_{g^-}(h)}(t)\right),\ \lambda_h\left(\beta_{\alpha_{h^-}(g)}(u)\right)\right)\\
&= \left(\alpha_{\beta_g(\beta_{g^-}(h))}\left(\lambda_{\alpha^{-1}_{\beta_g(\beta_{g^-}(h))}(g)}(t)\right),\ \beta_{\alpha_h(\alpha_{h^-}(g))}\left(\lambda_{\beta^{-1}_{\alpha_h(\alpha_{h^-}(g))}(h)}(u)\right)\right)\\
&= \left(\alpha_h\left(\lambda_{\alpha_h^{-1}(g)}(t)\right),\ \beta_g\left(\lambda_{\beta_g^{-1}(h)}\right)(u)\right),
\end{aligned}$$

for all $g,t \in G$ and $h,u \in H$, as described in Theorem 4.2 in [2].

3.4 A structural theorem for left semi-braces

As a consequence of Theorem 3.1.1, we may obtain any left semi-brace B as matched product of a skew left brace G and a trivial left semi-brace E.

Proposition 3.4.1. *Let B be a left semi-brace, E the set of idempotents of the semigroup $(B,+)$, 0 the identity of $(B,\circ)$ and $G = B + 0$. Then G is a skew left brace, E is a trivial semi-brace and B is isomorphic to the matched product of G and E.*

Proof. Let $H := G$ and $N := \{0\}$. Clearly H, N are normal subgroups of $(G,+)$, moreover, by Proposition 1.2.11, if $b \in B$, then $\rho_b(G) \subseteq G$ and $\lambda_b(0) \in \{0\} + E$. Obviously $G \cap \{0\} = \{0\}$ and $G + \{0\} = G$. Hence by Theorem 3.1.5 B is isomorphic to the matched product of the left semi-braces G and E. Finally, by Proposition 1.2.7, G is a skew left brace and E is a trivial left semi-brace. □

In the particular case of the matched product of a skew left brace G and a trivial semi-brace, requirements in Theorem 3.1.1 may be simplified as the following corollary shows.

Corollary 3.4.2. *Let G be a skew left brace and E a trivial left semi-brace $\delta : G \to \mathrm{Sym}(E)$ a right action of the group $(G,\circ)$ on the set E and $\sigma : E \to \mathrm{Aut}(G)$ a left action of the group $(E,\circ)$ on the set G and σ_e is an automorphism of the group $(G,+)$, for every $e \in E$, such that*

1. ${}^{e}(g_1 \circ g_2) = ({}^{e}g_1) \circ ({}^{e^{g_1}}g_2)$;
2. $(e_1 \circ e_2)^{g} = {e_1}^{{e_2}^{g}} \circ {e_2}^{g}$;
3. $0^{g} = 0$,

hold for all $g, g_1, g_2 \in G$ and $e, e_1, e_2 \in E$. Then the sum and the multiplication over the cartesian product $G \times E$ given by

$$(g_1, e_1) + (g_2, e_2) := (g_1 + g_2,\ e_2)$$
$$(g_1, e_1) \circ (g_2, e_2) := \left(g_1 \circ {\left((e_1^{-})^{g_1} \right)}^{-} g_2,\ e_1 \circ \left((e_2^{-})^{\left({e_1}^{-} g_1 \right)^{-}} \right)^{-} \right)$$

define a structure of left semi-brace.

Proof. Clearly conditions 1., 3. and 4. of Theorem 3.1.1 hold. Moreover condition 2. holds since $(G,+)$ is a group. Finally, if $e_1, e_2 \in E$ and $g \in G$, then

$$\left(\left((e_1+e_2)^-\right)^g\right)^- = \left(\left(e_2^-\right)^g\right)^- = \left(\left(e_1^-\right)^g\right)^- + \left(\left(e_2^-\right)^g\right)^-,$$

since $(E,+)$ is a right zero-semigroup. □

Example 3.4.3. Let B be the left semi-brace in Example 1.1.4. In this case $(B,\circ)$ is a group and there exists an endomorphism f of $(B,\circ)$ such that $f^2 = f$ and $a+b = b\circ f(a)$, for all $a,b \in B$. Moreover, if 0 is the identity of $(B,\circ)$, then $G := B+0 = \operatorname{Im} f$ and the set of idempotents E of $(B,+)$ is $\ker f$. As in Theorem 3.1.5, we may compute the right action $\delta : G \to \operatorname{Sym}(E)$ and the left action $\sigma : E \to \operatorname{Aut}(G)$. First, note that, if $g \in G$, then there exists $a \in B$, such that $g = f(a)$ and so $f(g) = f^2(a) = f(a) = g$, since $f^2 = f$. If $g \in G$ and $e \in E$, then

$$\delta_g(e) = \left(\lambda_{g^-}\left(e^-\right)\right)^- = \left(g^- \circ \left(g + e^-\right)\right)^- = \left(e^- \circ f(g)\right)^- \circ g = g^- \circ e \circ g$$

and

$$\sigma_e(g) = \left(\rho_{e^-}\left(g^-\right)\right)^- = \left(\left(g+e^-\right)^- \circ e^-\right)^- = e \circ \left(e^- \circ f(g)\right) = g.$$

4 Applications

4.1 Set-theoretical solutions of the Yang-Baxter equation

We recall the definition of set-theoretical solution of the quantum Yang-Baxter equation (see [15]) and provide some examples.

Definition 4.1.1. Let X be a set and $r : X \times X \to X \times X$. The pair (X, r) is said a *set-theoretical solution of the quantum Yang Baxter equation*, simply a *solution* if

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

is satisfied, where $r_1 := r \times \mathrm{id}_X$ and $r_2 := \mathrm{id}_X \times r$.

Example 4.1.2. Let X be a set and consider $\tau : X \times X \to X \times X$ the twist map, i.e., a map such that $r(x, y) = (y, x)$, for all $x, y \in X$. Then (X, τ) is a solution, called *trivial solution*.

Example 4.1.3 (see [18])**.** Let G be a group and let f an endomorphism of G such that $f^2 = f$. Then, the map $r : G \times G \to G \times G$ defined by

$$r(x, y) = \left(xyf(x)^{-1}, f(x)\right)$$

for all $x, y \in G$ is a solution.

In fact, if $x, y, z \in G$, then

$$\begin{aligned} r_1 r_2 r_1 (x, y, z) &= r_1 r_2 \left(xyf(x)^{-1}, f(x), z\right) \\ &= r_1 \left(xyf(x)^{-1}, f(x) zf(f(x))^{-1}, f(f(x))\right) \\ &= r_1 \left(xyf(x)^{-1}, f(x) zf(x)^{-1}, f(x)\right) \\ &= \left(xyf(x)^{-1} f(x) zf(x)^{-1} f\left(xyf(x)^{-1}\right)^{-1}, f\left(xyf(x)^{-1}\right), f(x)\right) \\ &= \left(xyzf(y)^{-1} f(x)^{-1}, f(x) f(y) f(x)^{-1}, f(x)\right) \end{aligned}$$

and

$$\begin{aligned} r_2r_1r_2\left(x,y,z\right) &= r_2r_1\left(x,yzf\left(y\right)^{-1},f\left(y\right)\right) \\ &= r_2\left(xyzf\left(y\right)^{-1}f\left(x\right)^{-1},f\left(x\right),f\left(y\right)\right) \\ &= \left(xyzf\left(y\right)^{-1}f\left(x\right)^{-1},f\left(x\right)f\left(y\right)f\left(f\left(x\right)\right)^{-1},f\left(f\left(x\right)\right)\right) \\ &= \left(xyzf\left(y\right)^{-1}f\left(x\right)^{-1},f\left(x\right)f\left(y\right)f\left(x\right)^{-1},f\left(x\right)\right). \end{aligned}$$

Note that we can find this kind of solutions in [18]. In particular, if $f=\mathrm{id}_G$ then r is the so-called Venkov's solution (see [15]).

Proposition 4.1.4. *Let X be a set and $r: X\times X\to X\times X$ a map. Set $r\left(x,y\right):=\left(\lambda_x\left(y\right),\rho_y\left(x\right)\right)$, for all $x,y\in X$. The pair $\left(X,r\right)$ is a solution if and only if*

1. $\lambda_x\lambda_y\left(z\right)=\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}\left(z\right)$,
2. $\rho_{\lambda_{\rho_y(x)}(z)}\left(\lambda_x\left(y\right)\right)=\lambda_{\rho_{\lambda_y(z)}(x)}\left(\rho_z\left(y\right)\right)$,
3. $\rho_z\rho_y\left(x\right)=\rho_{\rho_z(y)}\rho_{\lambda_y(z)}\left(x\right)$

hold, for all $x,y,z\in X$.

Proof. Let $x,y,z\in X$. Note that

$$\begin{aligned} r_1r_2r_1\left(x,y,z\right) &= r_1r_2\left(\lambda_x\left(y\right),\rho_y\left(x\right),z\right) \\ &= r_1\left(\lambda_x\left(y\right),\lambda_{\rho_y(x)}\left(z\right),\rho_z\rho_y\left(x\right)\right) \\ &= \left(\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}\left(z\right),\rho_{\lambda_{\rho_y(x)}(z)}\left(\lambda_x\left(y\right)\right),\rho_z\rho_y\left(x\right)\right) \end{aligned}$$

and

$$\begin{aligned} r_2r_1r_2\left(x,y,z\right) &= r_2r_1\left(x,\lambda_y\left(z\right),\rho_z\left(y\right)\right) \\ &= r_2\left(\lambda_x\lambda_y\left(z\right),\rho_{\lambda_y(z)}\left(x\right),\rho_z\left(y\right)\right) \\ &= \left(\lambda_x\lambda_y\left(z\right),\lambda_{\rho_{\lambda_y(z)}(x)}\left(\rho_z\left(y\right)\right),\rho_{\rho_z(y)}\rho_{\lambda_y(z)}\left(x\right)\right). \end{aligned}$$

In this way if r is a solution then *1., 2., 3.* hold for all $x,y,z\in X$ and conversely. □

Example 4.1.5 (see Example 2.8 in [23])**.** Let $\left(X,\vee,\wedge\right)$ be a distributive lattice and $r: X\times X\to X\times X$ defined by $r\left(x,y\right)=\left(x\wedge y,x\vee y\right)$. Then $\left(X,r\right)$ is a solution of the Yang-Baxter equation. In fact, if we set $\lambda_x\left(y\right):=x\wedge y$ and $\rho_y\left(x\right):=x\vee y$, then

$$\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}\left(z\right)=\lambda_{x\wedge y}\lambda_{x\vee y}\left(z\right)=\left(x\wedge y\right)\wedge\left(\left(x\vee y\right)\wedge z\right)=x\wedge y\wedge z=\lambda_x\lambda_y\left(z\right),$$

i.e., *1.* in the Proposition 4.1.4,

$$\begin{aligned}\rho_{\lambda_{\rho_y(x)}(z)}\left(\lambda_x\left(y\right)\right) &= \rho_{\lambda_{y\vee x}(z)}\left(x\wedge y\right) = \rho_{(y\vee x)\wedge z}\left(x\wedge y\right)\\ &= \left((y\vee x)\wedge z\right)\vee\left(x\wedge y\right) = \left(y\wedge z\right)\vee\left(x\wedge z\right)\vee\left(x\wedge y\right)\\ &= \left(y\vee z\right)\wedge\left(x\vee z\right)\wedge\left(x\vee y\right) = \left(\left(z\wedge y\right)\vee x\right)\wedge\left(z\vee y\right)\\ &= \lambda_{\rho(\lambda_z(y))(x)}\left(\rho_z\left(y\right)\right),\end{aligned}$$

that is *2.* in the same proposition and

$$\rho_{\rho_z(y)}\rho_{\lambda_y(z)}\left(x\right) = \rho_{z\vee y}\rho_{y\wedge z}\left(x\right) = \left(z\vee y\right)\vee\left(y\wedge z\right)\vee x = z\vee y\vee x = \rho_z\rho_y\left(x\right)$$

the last condition in Proposition 4.1.4. Therefore (X,r) is a solution.

Definition 4.1.6. Let (X,r) be a set-theoretical solution of quantum Yang-Baxter equation, with $r: X\times X\to X\times X$ such that $r\left(x,y\right) = \left(\lambda_x\left(y\right),\rho_y\left(x\right)\right)$. We say that (X,r) is

1. *right non-degenerate* if ρ_x is a bijection for every $x\in X$,
2. *left non-degenerate* if λ_x is a bijection for every $x\in X$,
3. *non-degenerate* if (X,r) is both right and left non-degenerate,
4. *involutive* if $r^2 = \mathrm{id}_{X\times X}$.

Examples 4.1.7. 1. The solution (X,τ) in Example 4.1.2 is non-degenerate and involutive.

2. The solution (G,r) in Example 4.1.3 is left non-degenerate. In fact let $x\in G$. If $y_1,y_2\in G$ such that $xy_1f\left(x\right)^{-1} = \lambda_x\left(y_1\right) = \lambda_x\left(y_2\right) = xy_2f\left(x\right)^{-1}$, then $y_1 = y_2$, since G is a group, moreover if $\bar{y}\in G$, setting $y := x^{-1}\bar{y}f\left(x\right)$, we have that $\lambda_x\left(y\right) = xx^{-1}\bar{y}f\left(x\right)f\left(x\right)^{-1} = \bar{y}$. Hence λ_x is bijective. Moreover (G,r) is right non-degenerate if and only if $f = \mathrm{id}_G$. Finally, (G,r) is involutive if and only if $f = \mathrm{id}_G$ and G is abelian.

3. The solution (X,r) in Example 4.1.5 in neither right nor left non-degenerate. Clearly it is not involutive.

4.2 Left non-degenerate solutions of the Yang-Baxter equation and left semi-braces

We may obtain a left non-degenerate solution of the Yang-Baxter equation from every left semi-brace, as done by Catino, Stefanelli and myself in [11].

Theorem 4.2.1. *Let B be a left semi-brace. Then, the function $r : B \times B \to B \times B$ given by*

$$r(a,b) = (a \circ (a^- + b),\ (a^- + b)^- \circ b), \tag{4.1}$$

for all $a, b \in B$, is a left non-degenerate solution of the Yang-Baxter equation that we call the solution associated to the left semi-brace B.

Proof. First we note that

$$a \circ b = a \circ (a^- + b) \circ (a^- + b)^- \circ b = \lambda_a(b) \circ \rho_b(a), \tag{4.2}$$

holds for all $a, b \in B$.
Now, let $x, y, z \in B$ and set

$$\begin{aligned}(t_1, t_2, t_3) &:= r_1 r_2 r_1 (x, y, z) \\ &= \left(\lambda_{\lambda_x(y)}\left(\lambda_{\rho_y(x)}(z)\right), \rho_{\lambda_{\rho_y(x)}(z)}\left(\lambda_x(y)\right), \rho_z\left(\rho_y(x)\right)\right)\end{aligned}$$

and

$$\begin{aligned}(s_1, s_2, s_3) &:= r_2 r_1 r_2 (x, y, z) \\ &= \left(\lambda_x\left(\lambda_y(z)\right), \lambda_{\rho_{\lambda_y(z)}(x)}\left(\rho_z(y)\right), \rho_{\rho_z(y)}\left(\rho_{\lambda_y(z)}(x)\right)\right).\end{aligned}$$

By relation (4.2), we have that

$$\begin{aligned}t_1 \circ t_2 \circ t_3 &= \lambda_{\lambda_x(y)}(\lambda_{\rho_y(x)}(z)) \circ \rho_{\lambda_{\rho_y(x)}(z)}(\lambda_x(y)) \circ \rho_z(\rho_y(x)) \\ &= \lambda_x(y) \circ \lambda_{\rho_y(x)}(z) \circ \rho_z(\rho_y(x)) \\ &= \lambda_x(y) \circ \rho_y(x) \circ z \\ &= x \circ y \circ z\end{aligned}$$

and similarly

$$\begin{aligned}s_1 \circ s_2 \circ s_3 &= \lambda_x(\lambda_y(z)) \circ \lambda_{\rho_{\lambda_y(z)}(x)}(\rho_z(y)) \circ \rho_{\rho_z(y)}(\rho_{\lambda_y(z)}(x)) \\ &= x \circ y \circ z.\end{aligned}$$

By Proposition 1.2.1 and relation (4.2), we have that

$$t_1 = \lambda_{\lambda_x(y)}(\lambda_{\rho_y(x)}(z)) = \lambda_{\lambda_x(y) \circ \rho_y(x)}(z) = \lambda_{x \circ y}(z) = \lambda_x(\lambda_y(z)) = s_1.$$

Moreover, by Proposition 1.2.9 and relation (4.2) we obtain

$$s_3 = \rho_{\rho_z(y)}(\rho_{\lambda_y(z)}(x)) = \rho_{\lambda_y(z) \circ \rho_z(y)}(x) = \rho_{y \circ z}(x) = \rho_z(\rho_y(x)) = t_3.$$

It follows that $t_2 = s_2$ and as a consequence $(t_1, t_2, t_3) = (s_1, s_2, s_3)$; therefore r is a solution of the Yang-Baxter equation. Furthermore, r is left non-degenerate, since λ_b is bijective, for every $b \in B$, by Proposition 1.2.1. □

Remark 4.2.2. If B is a left semi-brace and r is the solution associated to B, then

$$r(a,b) = (\lambda_a(b),\ \rho_b(a))\,,$$

for all $a, b \in B$, where λ_a and ρ_b are the functions introduced in propositions 1.2.1 and 1.2.9 respectively.
As direct consequence, if B is a left semi-brace that is not a skew left brace, then the solution r associated to B is not right non-degenerate. In fact, if E is the set of all idempotents of $(B, +)$, 0 the identity of $(B, \circ)$ and $G := B + 0$, then $|E| \geq 2$ and by Remark 1.2.12 we have that $\rho_a(B) = G$, for every $a \in B$ and so ρ_a functions are not bijective.

Example 4.2.3. Let B be the left semi-brace in the Example 1.1.4. Then, $\lambda_a(b) = a \circ (a^- + b) = a \circ b \circ f(a^-) = a \circ b \circ f(a)^-$ and $\rho_b(a) = (a^- + b)^- \circ b = f(a)$. Hence, the solution r associated to B is given by

$$r\,(a,b) = \left(a \circ b \circ f(a)^-,\ f(a)\right),$$

for all $a, b \in B$. Note that this solution is the one introduced in Example 4.1.3.
Moreover, if f is the null function, then $r(a,b) = (a \circ b, 0)$. Conversely if r is a solution associated to a left semi-brace such that $\rho_b(a) = 0$, for all $a, b \in B$, then B is a trivial left semi-brace and $r\,(a,b) = (a \circ b, 0)$, for all $a, b \in B$.
In fact, if a, b are in B, then $a + b = b \circ (\rho_b\,(a^-))^- = b \circ 0 = b$ and so $(B, +, \circ)$ is a trivial left semi-brace. Further, $\lambda_a(b) = a \circ (a^- + b) = a \circ b$, for all $a, b \in B$.

4.2.1 Left semi-braces as matched product and solutions of the Yang-Baxter equation

In Proposition 3.4.1 and Corollary 3.4.2, we prove that every left semi-brace B can be seen as the matched product of a skew left brace G and of a left trivial semi-brace E. Here we prove that the solution associated to the left semi-brace B may be obtained by solutions associated to G and E, respectively.

First we introduce a characterization of the matched product of a skew left brace and a left trivial semi-brace.

Lemma 4.2.4. *Let G be a skew left brace and E a trivial left semi-brace $\delta : G \to \mathrm{Sym}\,(E)$ a right action of the group $(G, \circ)$ on the set E and $\sigma : E \to \mathrm{Aut}\,(G)$ a left action of the group $(E, \circ)$ on the set G such that G, E, δ and σ define a matched*

product. Then there exist $\alpha : E \to \operatorname{Aut}(G)$ *a homomorphism from the group* $(E, \circ)$ *into the group of automorphisms of* $(G, +)$ *and* $\beta : G \to \operatorname{Sym}(E)$ *a homomorphism from the group* $(G, \circ)$ *into the symmetric group on* E*, such that*

$$\lambda_g \alpha_e = \alpha_{\beta_g(e)} \lambda_{\alpha^{-1}_{\beta_g(e)}(g)} \tag{4.3}$$

and

$$\lambda_e \beta_g = \beta_{\alpha_e(g)} \lambda_{\beta^{-1}_{\alpha_e(g)}(e)}, \tag{4.4}$$

for all $g \in G$ *and* $e \in E$*.*
Conversely, if G *is a skew left brace,* E *is a trivial left semi-brace,* $\alpha : E \to \operatorname{Aut}(G)$ *is a homomorphism from the group* $(E, \circ)$ *into the group of automorphisms of* $(G, +)$ *and* $\beta : G \to \operatorname{Sym}(E)$ *is a homomorphism from the group* $(G, \circ)$ *into the symmetric group on* E *that satisfy* (4.3) *and* (4.4)*, then there exist a right action* δ *of the group* $(G, \circ)$ *on the set* E *and a left action* $\sigma : E \to \operatorname{Aut}(G)$ *of the group* $(E, \circ)$ *on the set* G *that satisfy the assumptions of Corollary 3.4.2, i.e., that allow us to define the matched product of* G *and* E *by* δ *and* σ*.*

Proof. If $g \in G$ and $e \in E$, define

$$\alpha_e(g) := \sigma_e(g) \qquad \text{and} \qquad \beta_g(e) := \left(\delta_{g^-}\left(e^-\right)\right)^-,$$

then, clearly, α is a homomorphism from $(E, \circ)$ into the group of automorphism of $(G, +)$. Further, if $g_1, g_2 \in G$ and $e \in E$, then

$$\begin{aligned}\beta_{g_1}(\beta_{g_2}(e)) &= \left(\delta_{g_1^-}\left(\delta_{g_2^-}\left(e^-\right)\right)\right)^- = \left(\delta_{g_2^- \circ g_1^-}\left(e^-\right)\right)^- = \left(\delta_{(g_1 \circ g_2)^-}\left(e^-\right)\right)^- \\ &= \beta_{g_1 \circ g_2}(e),\end{aligned}$$

since δ is a right action, and

$$\beta_0(e) = \left(\delta_0\left(e^-\right)\right)^- = e = \operatorname{id}_E(e).$$

Hence $\beta_g \in \operatorname{Sym}(E)$, for every $e \in E$ and β is a homomorphism from the group $(G, \circ)$ into the symmetric group on E. Note that, if $g \in G$ and $e \in E$, then

$$\left(\left(\left(e^-\right)^{g^-}\right)^-\right)^{\left(\left(e^-\right)^{g^-} g\right)} = \left(e^{e^-\left(g^-\right)}\right)^{\left(\left(\left(e^-\right)_{g^-}\right)^-\right)} = e$$

and

$${}^{e}\left(\left(\left(e^-\right)^{g^-} g\right)^-\right) = {}^{e}\left(e^- g^-\right) = g^-.$$

Hence, if $g, t \in G$ and $e \in E$, then

$$\alpha_{\beta_g(e)}\lambda_{\alpha^{-1}_{\beta_g(e)}(g)}(t) = \left(\left(e^{-}\right)^{g^{-}}\right)^{-}\left(\left(e^{-}\right)^{g^{-}} g \circ \left(\left(\left(e^{-}\right)^{g^{-}} g\right)^{-} + t\right)\right)$$

$$= \left(\left(\left(e^{-}\right)^{g^{-}}\right)^{-}\left(\left(e^{-}\right)^{g^{-}} g\right)\right) \circ \left(\left(\left(\left(e^{-}\right)^{g^{-}}\right)^{-}\right)^{\left(\left(e^{-}\right)^{g^{-}} g\right)}\left(\left(\left(e^{-}\right)^{g^{-}} g\right)^{-} + t\right)\right)$$

$$= g \circ \left(g^{-} + {}^{e}t\right) = \lambda_g\left(\alpha_e(t)\right),$$

i.e., condition (4.3) is satisfied. Finally, if $g \in G$ and $e, u \in E$, then

$$\beta_{\alpha_e(g)}\lambda_{\beta^{-1}_{\alpha_e(g)}(e)}(u) = \left(\left(\left(\left(\left(e^{-}\right)^{{}^{e}g}\right)^{-} \circ u\right)^{-}\right)^{(^{e}g)^{-}}\right)^{-} = \left(\left(u^{-} \circ \left(e^{g}\right)^{-}\right)^{(^{e}g)^{-}}\right)^{-}$$

$$= \left(\left(u^{-}\right)^{\left(e^{g}\right)^{-}({}^{e}g)^{-}} \circ \left(\left(e^{g}\right)^{-}\right)^{(^{e}g)^{-}}\right)^{-} = \left(\left(u^{-}\right)^{\left(e^{g}\right)^{-}\left({}^{e}g^{-}\right)} \circ \left(\left(e^{-}\right)^{{}^{e}g}\right)^{(^{e}g)^{-}}\right)^{-}$$

$$= e \circ \left(\left(u^{-}\right)^{g^{-}}\right)^{-} = \lambda_e\left(\beta_g(u)\right),$$

i.e., condition (4.4) holds.
On the other hand, define

$$\sigma_e(g) := \alpha_e(g) \qquad \text{and} \qquad \delta_g(e) := \beta^{-1}_{\alpha_e(g)}(e),$$

for all $e \in E$ and $g \in G$. Clearly, $\sigma : E \to \mathrm{Aut}(G)$ is a left action of the group $(E, \circ)$ on the set G, such that σ_e lies in the group of automorphism of $(G, +)$. Further, if $g_1, g_2 \in G$ and $e \in E$, then

$$\delta_{g_1 \circ g_2}(e) = \beta^{-1}_{\alpha_e(g_1 \circ g_2)}(e) = \beta^{-1}_{\alpha_e\left(g_1 + \lambda_{g_1}(g_2)\right)}(e) = \beta^{-1}_{\left(\alpha_e(g_1) + \alpha_e\left(\lambda_{g_1}(g_2)\right)\right)}(e)$$

$$= \beta^{-1}_{\left(\alpha_e(g_1) + \lambda_{\alpha_e(g_1)}\left(\alpha_{\beta^{-1}_{\alpha_e(g_1)}(e)}(g_2)\right)\right)}(e) = \beta^{-1}_{\alpha_e(g_1) \circ \left(\alpha_{\beta^{-1}_{\alpha_e(g_1)}(e)}(g_2)\right)}(e)$$

$$= \beta_{\left(\alpha_e(g_1) \circ \left(\alpha_{\beta^{-1}_{\alpha_e(g_1)}(e)}(g_2)\right)\right)^{-}}(e) = \beta^{-1}_{\alpha_{\beta^{-1}_{\alpha_e(g_1)}(e)}(g_2)}\beta^{-1}_{\alpha_e(g_1)}(e)$$

$$= \delta_{g_2}\delta_{g_1}(e)$$

and

$$\delta_0(e) = \beta^{-1}_{\alpha_e(0)}(e) = e,$$

since $\alpha_e(0) = 0$. Hence δ is a right action from the group $(G, \circ)$ on the set E. If $e \in E$ and $g_1, g_2 \in G$, then

$$\begin{aligned}
&{}^{e}(g_1 \circ g_2) = \sigma_e(g_1 \circ g_2) = \alpha_e(g_1 + \lambda_{g_1}(g_2)) = \alpha_e(g_1) + \alpha_e(\lambda_{g_1}(g_2)) \\
&= \alpha_e(g_1) + \lambda_{\alpha_e(g_1)}\alpha_{\beta^{-1}_{\alpha_e(g_1)}(e)}(g_2) = (\alpha_e(g_1)) \circ \left(\alpha_{\beta^{-1}_{\alpha_e(g_1)}(e)}(g_2)\right) \\
&= \sigma_e(g_1) \circ \sigma_{\delta_{g_1}(e)}(g_2) = {}^{e}g_1 \circ {}^{e^{g_1}}g_2,
\end{aligned}$$

i.e., condition 1. in Corollary 3.4.2 holds. If $e_1, e_2 \in E$ and $g \in G$, then

$$\begin{aligned}
&(e_1 \circ e_2)^g = \delta_g(e_1 \circ e_2) = \beta^{-1}_{\alpha_{e_1 \circ e_2}(g)}(e_1 + \lambda_{e_1}(e_2)) = \beta^{-1}_{\alpha_{e_1 \circ e_2}(g)}(e_1) \\
&+ \beta^{-1}_{\alpha_{e_1 \circ e_2}(g)}(\lambda_{e_1}(e_2)) = \beta^{-1}_{\alpha_{e_1 \circ e_2}(g)}(e_1) + \lambda_{\beta^{-1}_{\alpha_{e_1 \circ e_2}(g)}(e_1)}\beta^{-1}_{\alpha^{-1}_{e_1}(\alpha_{e_1 \circ e_2}(g))}(e_2) \\
&= \beta^{-1}_{\alpha_{e_1}(\alpha_{e_2}(g))}(e_1) \circ \beta^{-1}_{\alpha_{e_2}(g)}(e_2) = \delta_{\sigma_{e_2}(g)}(e_1) \circ \delta_g(e_2) = e_1^{{}^{e_2}g} \circ e_2^g
\end{aligned}$$

i.e., condition 2. in Corollary 3.4.2 holds. Moreover note that from (4.4) it follows $\lambda_{\beta^{-1}_{\alpha_0(g)}(0)} = \beta^{-1}_{\alpha_0(g)}\lambda_0\beta_g$, i.e., $\lambda_{\beta_g^{-1}(0)} = \beta_g^{-1}\beta_g = \mathrm{id}_E$, for every $g \in G$, then

$$0 = \lambda_{\beta_g^{-1}(0)}(0) = \beta_g^{-1}(0) \circ \left(\left(\beta_g^{-1}(0)\right)^{-} + 0\right) = \beta_g^{-1}(0) \circ 0 = \beta_g^{-1}(0) = \delta_g(0) = 0^g,$$

for every $g \in G$ and so condition 3. in Corollary 3.4.2 holds. Therefore G, E, σ and δ satisfy assumption of Corollary 3.4.2 and we may construct the matched product of G and E, by σ and δ. □

Now we write the solution of the Yang-Baxter equation associated to the left semi-brace $B := G \bowtie E$ in terms of α, β, λ_g, λ_e and ρ_g functions.

Proposition 4.2.5. *Let $B := G \bowtie E$ be the left semi-braces obtained as the matched product of G and E by α and β as in Lemma 4.2.4. Then, the solution of the Yang-Baxter equation associated to B is the map $r : B \times B \to B \times B$ given by*

$$\begin{aligned}
r((g_1, e_1), (g_2, e_2)) = \Big(&\left(\lambda_{g_1}\left(\alpha_{\beta_{g_1}^{-1}(e_1)}(g_2)\right),\ \lambda_{e_1}\left(\beta_{\alpha_{e_1}^{-1}(g_1)}(e_2)\right)\right), \\
&\left(\rho_{\alpha_{e_2}^{-1}(g_2)}\left(\alpha^{-1}_{\lambda_{e_1}\left(\beta_{\alpha_{e_1}^{-1}(g_1)}(e_2)\right)}(g_1)\right),\ 0\right)\Big),
\end{aligned}$$

for all $e_1, e_2 \in E$ and $g_1, g_2 \in G$.

Proof. According to Theorem 4.2.1, we have to compute $\lambda_{(g_1,e_1)}$ functions and $\rho_{(g_2,e_2)}$ ones.

If $(g_1, e_1), (g_1, e_1) \in G \times E$, then

$$\begin{aligned}
\lambda_{(g_1,e_1)}(g_2, e_2) &= (g_1, e_1) \circ \left((g_1, e_1)^- + (g_2, e_2) \right) \\
&= (g_1, e_1) \circ \left(\left(\left({e_1}^- g_1 \right)^-, ({e_1}^-)^{g_1} \right) + (g_2, e_2) \right) \\
&= (g_1, e_1) \circ \left(\left({e_1}^- g_1 \right)^- + g_2, e_2 \right) \\
&= \left(g_1 \circ ((e_1^-)^{g_1})^- \left(((e_1^-)^{g_1}) {g_1}^- + g_2 \right),\ e_1 \circ \left((e_2^-)^{\left({e_1}^- g_1 \right)^-} \right)^- \right) \\
&= \left(g_1 \circ \left(g_1^- + ((e_1^-)^{g_1})^- g_2 \right),\ e_1 \circ \left({e_1}^- + \left((e_2^-)^{\left({e_1}^- g_1 \right)^-} \right)^- \right) \right) \\
&= \left(\lambda_{g_1} \left(\alpha_{\beta_{g_1^-}(e_1)}(g_2) \right),\ \lambda_{e_1} \left(\beta_{\alpha_{e_1^-}(g_1)}(e_2) \right) \right) \\
&= \left(\lambda_{g_1} \left(\alpha_{\beta_{g_1}^{-1}(e_1)}(g_2) \right),\ \lambda_{e_1} \left(\beta_{\alpha_{e_1}^{-1}(g_1)}(e_2) \right) \right).
\end{aligned}$$

Further, if $(g_1, e_1), (g_1, e_1) \in G \times E$, then

$$\rho_{(g_2,e_2)}(g_1, e_1) = \left((g_1, e_1)^- + (g_2, e_2) \right)^- \circ (g_2, e_2) = \left(\left({e_1}^- g_1 \right)^- + g_2, e_2 \right)^- \circ (g_2, e_2)$$

$$= \left(\left({e_2}^- \left(\left({e_1}^- g_1 \right)^- + g_2 \right) \right)^-, (e_2^-)^{\left(\left({e_1}^- g_1 \right)^- + g_2 \right)} \right) \circ (g_2, e_2).$$

The second component of $\rho_{(g_2,e_2)}(g_1, e_1)$ is given by

$$(e_2^-)^{\left(\left({e_1}^- g_1 \right)^- + g_2 \right)} \circ \left((e_2^-)^{\left(\left(\left({e_2}^- \right)^{\left(\left({e_1}^- g_1 \right)^- + g_2 \right)} \right)^- \left({e_2}^- \left(\left({e_1}^- g_1 \right)^- + g_2 \right) \right)^- \right)^-} \right)^- = 0,$$

since $f^{\left((f^x)^- (f x)^- \right)^-} = f^{\left((f^x)^- (f^x x^-) \right)^-} = f^x$, for all $f \in E$ and $x \in g$ and we take $f := e_2^-$ and $x := \left({e_1}^- g_1 \right)^- + g_2$.

Finally, the first component of $\rho_{(g_2,e_2)}(g_1,e_1)$ is given by

$$\begin{aligned}
&\left(e_2{}^{-}\left(\left(e_1{}^{-}g_1\right)^{-}+g_2\right)\right)^{-}\circ\left(\left(\left(e_2{}^{-}\right)^{\left(\left(e_1{}^{-}g_1\right)^{-}+g_2\right)}\right)^{-\left(e_2{}^{-}\left(\left(e_1{}^{-}g_1\right)^{-}+g_2\right)\right)^{-}}\right)^{-}g_2\\
&=\left(e_2{}^{-}\left(\left(e_1{}^{-}g_1\right)^{-}+g_2\right)\right)^{-}\circ e_2{}^{-}g_2=\rho_{\left(e_2{}^{-}g_2\right)}\left(\left(e_2{}^{-}\left(e_1{}^{-}g_1\right)^{-}\right)^{-}\right)\\
&=\rho_{\alpha_{e_2^{-}}(g_2)}\left(\left(\alpha_{e_2-}\left(\left(e_1{}^{-}g_1\right)^{-}\right)\right)^{-}\right)=\rho_{\alpha_{e_2}^{-1}(g_2)}\left(\alpha^{-1}_{\beta_{\left(e_1{}^{-}g_1\right)}(e_2)}\left(e_1{}^{-}g_1\right)\right)\\
&=\rho_{\alpha_{e_2}^{-1}(g_2)}\left(\alpha^{-1}_{\beta_{\alpha_{e_1{}^{-}}(g_1)}(e_2)}\left(\alpha_{e_1{}^{-}}(g_1)\right)\right)=\rho_{\alpha_{e_2}^{-1}(g_2)}\left(\alpha^{-1}_{\beta_{\alpha^{-1}_{e_1}(g_1)}(e_2)}\left(\alpha^{-1}_{e_1}(g_1)\right)\right)\\
&=\rho_{\alpha_{e_2}^{-1}(g_2)}\left(\alpha^{-1}_{e_1\circ\beta_{\alpha^{-1}_{e_1}(g_1)}(e_2)}(g_1)\right)=\rho_{\alpha_{e_2}^{-1}(g_2)}\left(\alpha^{-1}_{e_1\circ\left(e_1{}^{-}+\beta_{\alpha^{-1}_{e_1}(g_1)}(e_2)\right)}(g_1)\right)\\
&=\rho_{\alpha_{e_2}^{-1}(g_2)}\left(\alpha^{-1}_{\lambda_{e_1}\left(\beta_{\alpha^{-1}_{e_1}(g_1)}(e_2)\right)}(g_1)\right),
\end{aligned}$$

Where for second equality we use the fact that

$$\left(\left(f^{x}\right)^{-}\right)^{\left(fx\right)^{-}}=\left(\left(f^{-}\right)^{fx}\right)^{\left(fx\right)^{-}}=f^{-},$$

for all $x\in G$ and $f\in E$ with $f:=e_2^{-}$ and $x:=\left(e_1^{-}g_1\right)^{-}+g_2$. □

If B is a left semi-brace, 0 is the identity of $(B,\circ)$, $G:=B+0$ and E is the set of idempotents then, by Proposition 3.4.1, B may be seen as the matched product of the skew left brace G and the trivial semi-brace E. The isomorphism ψ from $G\bowtie E$ into B is given by $\psi(g,e):=g+e$, for all $g\in G$ and $e\in E$. Hence the solution associated to B is, by Proposition 4.2.5, given by

$$\lambda_{(g_1+e_1)}(g_2+e_2)=\lambda_{g_1}\left(\alpha_{\beta^{-1}_{g_1}(e_1)}(g_2)\right)+\lambda_{e_1}\left(\beta_{\alpha^{-1}_{e_1}(g_1)}(e_2)\right)$$

and

$$\rho_{(g_2+e_2)}(g_1+e_1)=\rho_{\alpha_{e_2}^{-1}(g_2)}\left(\alpha^{-1}_{\lambda_{e_1}\left(\beta_{\alpha^{-1}_{e_1}(g_1)}(e_2)\right)}(g_1)\right)+0,$$

for all $g_1+e_1,g_2+e_2\in B$, where according with Lemma 4.2.4 and Theorem 3.1.5, α and β are

$$\alpha_e(g)=\sigma_e(g)=\left(\rho_{e^{-}}\left(g^{-}\right)\right)^{-}$$

and

$$\beta_g(e) = \left(\delta_{g^-}\left(e^-\right)\right)^- = \lambda_g(e),$$

for all $g \in G$ and $e \in E$.

Example 4.2.6. Let B be the left semi-brace in Example 1.1.4. In this case $(B, \circ)$ is a group and there exists an endomorphism f of $(B, \circ)$ such that $f^2 = f$ and $a + b = b \circ f(a)$, for all $a, b \in B$. Moreover, if 0 is the identity of $(B, \circ)$, then $G := B + 0 = \operatorname{Im} f$ and the set E of idempotents of $(B, +)$ is $\ker f$. In Example 3.4.3 we determine the actions δ and σ to obtain the matched product, so we may compute α and β to see the solution associated to B as solution associated to the matched product of G and E. If $g \in G$ and $e \in E$, then

$$\alpha_e(g) = {}^e g = g$$

and

$$\beta_g(e) = \left(\left(e^-\right)^{g^-}\right)^- = \left(g \circ e^- \circ g^-\right)^- = g \circ e \circ g^- = \delta_{g^-}(e).$$

Then the solution $r : (G \times E) \times (G \times E) \to (G \times E) \times (G \times E)$ associated to the left semi-brace $G \bowtie E$ is given by $r((g_1, e_1), (g_2, e_2)) = \left(\lambda_{(g_1,e_1)}(g_2, e_2), \rho_{(g_2,e_2)}(g_1, e_1)\right)$, where

$$\begin{aligned}\lambda_{(g_1,e_1)}(g_2, e_2) &= \left(\lambda_{g_1}\left(\alpha_{\beta_{g_1}^{-1}(e_1)}(g_2)\right),\ \lambda_{e_1}\left(\beta_{\alpha_{e_1}^{-1}(g_1)}(e_2)\right)\right) \\ &= \left(\lambda_{g_1}(g_2),\ \lambda_{e_1}(\beta_{g_1}(e_2))\right) = \left(\lambda_{g_1}(g_2),\ \lambda_{e_1}\left(e_2^{g_1^-}\right)\right)\end{aligned}$$

and

$$\rho_{(g_2,e_2)}(g_1, e_1) = \left(\rho_{\alpha_{e_2}^{-1}(g_2)}\left(\alpha^{-1}_{\lambda_{e_1}\left(\beta_{\alpha_{e_1}^{-1}(g_1)}(e_2)\right)}(g_1)\right),\ 0\right) = \left(\rho_{g_2}(g_1),\ 0\right),$$

for all $g_1, g_2 \in G$ and $e_1, e_2 \in E$. Hence

$$r((g_1, e_1), (g_2, e_2)) = \left(\left(\lambda_{g_1}(g_2),\ \lambda_{e_1}\left(e_2^{g_1^-}\right)\right),\ \left(\rho_{g_2}(g_1),\ 0\right)\right)$$

for all $g_1, g_2 \in G$ and $e_1, e_2 \in E$.

4.3 Holomorph of a right group and left semi-braces

In this section we introduce a suitable definition of the holomorph of a right group and we provide a description of left semi-braces in terms of regular subgroups of

the holomorph of a fixed right group (to appear in [8]). Notice that this result generalizes the analogue one given by Guarnieri and Vendramin in [19].

Definition 4.3.1. Let $(B,+)$ be a right group. The *holomorph of the right group* B is the semigroup $\mathrm{Hol}(B) := B \times \mathrm{Aut}(B)$, where the product is given by

$$(a,\alpha)(b,\beta) := (a+\alpha(b),\ \alpha\beta),$$

for all $a,b \in B$ and $\alpha,\beta \in \mathrm{Aut}(B)$.

The following proposition describes some basic properties of the holomorph of a right group.

Proposition 4.3.2. *Let B be a right group and $\mathrm{Hol}(B)$ the holomorph of B.*

1. *$\mathrm{Hol}(B)$ is a right group.*
2. *If E is the set of all idempotents of B, $\bar{e} \in E$ fixed and $G := B + \bar{e}$, then $\mathrm{Aut}(B) = \mathrm{Aut}(G) \times \mathrm{Sym}(E)$.*

Proof. 1. If $a,b,c \in B$ and $\alpha,\beta,\gamma \in \mathrm{Aut}(B)$ such that $(a,\alpha)(b,\beta) = (a,\alpha)(c,\gamma)$, then

$$\begin{cases} a+\alpha(b) = a+\alpha(c) \\ \alpha\beta = \alpha\gamma \end{cases}$$

and, since $(B,+)$ is left cancellative semigroup and α is a bijection, we have that $b=c$ and $\beta=\gamma$. Hence $\mathrm{Hol}(B)$ is left cancellative. Moreover if $a,c \in B$ and $\alpha,\gamma \in \mathrm{Aut}(B)$, then there exists $y \in B$ such that $a+y=c$. Set $x := \alpha^{-1}(y)$ and $\chi := \alpha^{-1}\gamma$, then

$$(a,\alpha)(x,\chi) = (a+\alpha(x),\alpha\chi) = \left(a+\alpha\alpha^{-1}(y),\alpha\alpha^{-1}\gamma\right) = (a+y,\gamma) = (c,\gamma).$$

Therefore, by Definition 1.2.2, $\mathrm{Hol}(B)$ is a right group.
2. Let $\gamma \in \mathrm{Aut}(B)$. First, note that $\left(\gamma_{|_E}\right)_{\#}$ the right and left reduction of map γ belongs to $\mathrm{Sym}(E)$. In fact, if $e \in E$, then $\gamma(e)+\gamma(e) = \gamma(e+e) = \gamma(e)$, i.e., $\gamma(e) \in E$; conversely if $e \in E$, then, since $\gamma \in \mathrm{Aut}(B)$, there exists $b \in B$ such that $\gamma(b) = e$ moreover $b \in E$ since $\gamma(b) = e = e+e = \gamma(b)+\gamma(b) = \gamma(b+b)$, i.e.,

$b = b + b$. Set $\bar{\gamma} : G \to G$ defined by $\bar{\gamma}(b + \bar{e}) = \gamma(b) + \bar{e}$. The map $\bar{\gamma}$ is injective. In fact, if $b_1, b_2 \in B$ such that $\bar{\gamma}(b_1 + \bar{e}) = \bar{\gamma}(b_2 + \bar{e})$, then

$$\begin{aligned} \gamma(b_1) + \bar{e} = \gamma(b_2) + \bar{e} &\implies \gamma\left(b_1 + \gamma^{-1}(\bar{e})\right) = \gamma\left(b_2 + \gamma^{-1}(\bar{e})\right) \\ &\implies b_1 + \gamma^{-1}(\bar{e}) = b_2 + \gamma^{-1}(\bar{e}) \\ &\implies b_1 + \gamma^{-1}(\bar{e}) + \bar{e} = b_2 + \gamma^{-1}(\bar{e}) + \bar{e} \\ &\implies b_1 + \bar{e} = b_2 + \bar{e}, \end{aligned}$$

since γ is bijective and $\gamma^{-1}(\bar{e}) \in E$ is a left identity. The map $\bar{\gamma}$ is surjective. In fact, if $b + \bar{e} \in G$, then there exists $a \in B$ such that $\gamma(a) = b + \bar{e}$ and so

$$\bar{\gamma}(a + \bar{e}) = \gamma(a) + \bar{e} = b + \bar{e}.$$

The map $\bar{\gamma}$ is an homomorphism. In fact, if $b_1, b_2 \in B$, then

$$\begin{aligned} \bar{\gamma}((b_1 + \bar{e}) + (b_2 + \bar{e})) = \bar{\gamma}((b_1 + b_2) + \bar{e}) = \gamma(b_1 + b_2) + \bar{e} = \gamma(b_1) + \gamma(b_2) + \bar{e} \\ = \gamma(b_1) + \bar{e} + \gamma(b_2) + \bar{e} = \bar{\gamma}(b_1 + \bar{e}) + \bar{\gamma}(b_2 + \bar{e}), \end{aligned}$$

since $\bar{e} \in E$. Therefore $\bar{\gamma} \in \mathrm{Aut}(G)$.
Consider the map $\varphi : \mathrm{Aut}(B) \to \mathrm{Aut}(G) \times \mathrm{Sym}(E)$ defined by $\varphi(\gamma) = \left(\bar{\gamma}, \left(\gamma_{|_E}\right)_{\#}\right)$. The map φ is injective. In fact, if $\gamma_1, \gamma_2 \in \mathrm{Aut}(B)$ such that $\varphi(\gamma_1) = \varphi(\gamma_2)$, then $\bar{\gamma}_1 = \bar{\gamma}_2$ and $\gamma_{1_{|_E}} = \gamma_{2_{|_E}}$. Hence, since if $b \in B$, then there exists $e \in E$ such that $b + e = b$ and

$$\begin{aligned} \gamma_1(b) = \gamma_1(b + e) = \gamma_1(b) + \gamma_1(e) = \gamma_1(b) + \bar{e} + \gamma_1(e) = \bar{\gamma}_1(b + \bar{e}) + \gamma_{1_{|_E}}(e) \\ = \bar{\gamma}_2(b + \bar{e}) + \gamma_{2_{|_E}}(e) = \gamma_2(b) + \bar{e} + \gamma_2(e) = \gamma_2(b) + \gamma_2(e) \\ = \gamma_2(b + e) = \gamma_2(b), \end{aligned}$$

i.e., $\gamma_1 = \gamma_2$. The map φ is surjective. In fact, if $(\alpha, \sigma) \in \mathrm{Aut}(G) \times \mathrm{Sym}(E)$, set $\gamma : B \to B$ such that $\gamma(b) = \alpha(g) + \sigma(e)$, for every $b = g + e \in B$. If $b_1 = g_1 + e_1, b_2 = g_2 + e_2 \in B$ such that $\gamma(b_1) = \gamma(b_2)$, then

$$\begin{aligned} \gamma(g_1 + e_1) = \gamma(g_2 + e_2) &\implies \alpha(g_1) + \sigma(e_1) = \alpha(g_2) + \sigma(e_2) \\ &\implies \alpha(g_1) = \alpha(g_2) \text{ and } \sigma(e_1) = \sigma(e_2) \\ &\implies g_1 = g_2 \text{ and } e_1 = e_2, \end{aligned}$$

i.e., γ is injective. Moreover if $b = g + e \in B$, then there exist $g_1 \in G$ and $e_1 \in E$, such that $\alpha(g_1) = g$ and $\sigma(e_1) = e$, so $\gamma(g_1 + e_1) = \alpha(g_1) + \sigma(e_1) = g + e = b$. Hence γ is surjective. Finally, if $b_1 = g_1 + e_1, b_2 = g_2 + e_2 \in B$, then

$$\begin{aligned}\gamma(b_1 + b_2) &= \gamma(g_1 + e_1 + g_2 + e_2) = \gamma(g_1 + g_2 + e_2) = \alpha(g_1 + g_2) + \sigma(e_2) \\ &= \alpha(g_1) + \alpha(g_2) + \sigma(e_2) = \alpha(g_1) + \sigma(e_1) + \alpha(g_2) + \sigma(e_2) \\ &= \gamma(b_1) + \gamma(b_2),\end{aligned}$$

i.e., γ is a homomorphism of B. Therefore φ is surjective. Finally, φ is a homomorphism from $\mathrm{Aut}(B)$ into $\mathrm{Aut}(G) \times \mathrm{Sym}(E)$. In fact, if $\gamma_1, \gamma_2 \in \mathrm{Aut}(B)$ and $g \in G$, then, clearly $\gamma_{1_{|E}}\gamma_{2_{|E}} = (\gamma_1\gamma_2)_{|E}$ and

$$\bar{\gamma}_1(\bar{\gamma}_2(g)) = \bar{\gamma}_1(\gamma_2(g) + \bar{e}) = \gamma_1(\gamma_2(g)) + \bar{e} = \overline{\gamma_1\gamma_2}(g),$$

so $\overline{\gamma_1\gamma_2} = \bar{\gamma}_1\bar{\gamma}_2$. Hence $\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1)\varphi(\gamma_2)$. Therefore φ is a group isomorphism. □

Let $\mathrm{pr}_1 : \mathrm{Hol}(B) \to B, (a, \alpha) \mapsto a$ be the first projection. Then every subgroup N of $\mathrm{Hol}(B)$ acts on B in the following way, if $(a, \alpha) \in N$ and $x \in B$ then

$$(a, \alpha) \cdot x = \mathrm{pr}_1((a, \alpha)(x, \mathrm{id}_B)) = a + \alpha(x).$$

Definition 4.3.3. Let B be a right group, $\mathrm{Hol}(B)$ its holomorph and N a subgroup of $\mathrm{Hol}(B)$. We say that N is a *regular* subgroup if for all $a, b \in B$ there exists a unique $(x, \chi) \in N$ such that

$$(x, \chi) \cdot a = b.$$

We characterize regular subgroups of the holomorph of a right group in the following way.

Lemma 4.3.4. *Let B be a right group, $\mathrm{Hol}(B)$ its holomorph. A subgroup N of $\mathrm{Hol}(B)$ is regular if and only if there exists a map $\phi : B \to \mathrm{Aut}(B)$ such that, setting $\phi_a := \phi(a)$, we have $\phi_a\phi_b = \phi_{a+\phi_a(b)}$ and $N = \{(a, \phi_a) \mid a \in B\}$.*

Proof. If (e, ε) is the identity of N, then

$$(a, \alpha)(e, \varepsilon) = (a, \alpha) \implies \begin{cases} a + \alpha(e) = a \\ \alpha\varepsilon = \alpha \end{cases} \tag{4.5}$$

and

$$(e,\varepsilon)(a,\alpha)=(a,\alpha) \quad \Longrightarrow \quad \begin{cases} e+\varepsilon(a)=a \\ \varepsilon\alpha=\alpha \end{cases}$$

so e belongs to the set E of all idempotents of B and $\varepsilon=\mathrm{id}_B$. Moreover, since N is regular, there exists a unique $(c,\gamma)\in N$ such that $(c,\gamma)\cdot e=b$, for every $b\in B$, then $c+\gamma(e)=b$, i.e., by (4.5), $c=b$. If $b\in B$, set $\phi_b:=\gamma$, where γ is the unique element of $\mathrm{Aut}(B)$ such that $(b,\gamma)\in N$ and $(b,\gamma)\cdot e=b$. In this way, $N=\{(a,\phi_a)\mid a\in B\}$. Moreover, if $a,b\in B$, then

$$(a,\phi_a)(b,\phi_b)=(a+\phi_a(b),\phi_a\phi_b),$$

since N is a group and by regularity of N, we have that $\phi_a\phi_b=\phi_{a+\phi_a(b)}$. Conversely, let (e,ϕ_e) be the identity of N. Then

$$(e,\phi_e)(e,\phi_e)=(e,\phi_e) \quad \Longrightarrow \quad \begin{cases} e+\phi_e(e)=e \\ \phi_e\phi_e=\phi_e \end{cases} \quad \Longrightarrow \quad \begin{cases} e+e=e \\ \phi_e=\mathrm{id}_B \end{cases}$$

so e belongs to the set E of all idempotents of B and $\phi_e=\mathrm{id}_B$. Moreover

$$(a,\phi_a)(e,\phi_e)=(a,\phi_a) \quad \Longrightarrow \quad a+\phi_a(e)=a. \tag{4.6}$$

Since N is a group for every $(a,\phi_a)\in N$ there exists (b,ϕ_b) such that $(a,\phi_a)(b,\phi_b)=(e,\mathrm{id}_B)=(b,\phi_b)(a,\phi_a)$, i.e.,

$$\begin{cases} a+\phi_a(b)=e=b+\phi_b(a) \\ \phi_a\phi_b=\mathrm{id}_B=\phi_b\phi_a \end{cases} \tag{4.7}$$

we indicate with a^- the unique element $b\in B$ such that (b,ϕ_b) satisfies (4.7). If $c,d\in B$, then by (4.6) and (4.7)

$$\begin{aligned} \left(d+\phi_d(c^-),\phi_{d+\phi_d(c^-)}\right)\cdot c &= (d+\phi_d(c^-),\phi_d\phi_{c^-})\cdot c=(d,\phi_d)(c^-,\phi_{c^-})\cdot c \\ &= (d,\phi_d)\cdot(c^-+\phi_{c^-}(c))=(d,\phi_d)\cdot e=d. \end{aligned}$$

Hence N is a regular subgroup of $\mathrm{Hol}(B)$. □

Note that if B is a right group and N is a regular subgroup of $\mathrm{Hol}(B)$, φ is an automorphism of $(B,+)$ and e the idempotent of B such that (e,ϕ_e) is the identity

of N, then also $N_\varphi := (e,\varphi)\, N\, (e,\varphi^{-1})$ is a regular subgroup of $\mathrm{Hol}\,(B)$. In fact, clearly N_φ is a group. Further if $a \in B$, then

$$\begin{aligned}(e,\varphi)\,(a,\phi_a)\left(e,\varphi^{-1}\right) &= (e+\varphi(a),\varphi\phi_a)\left(e,\varphi^{-1}\right) = (\varphi(a),\varphi\phi_a)\left(e,\varphi^{-1}\right)\\ &= \left(\varphi(a)+\varphi\phi_a(e),\varphi\phi_a\varphi^{-1}\right) = \left(\varphi(a+\phi_a(e)),\varphi\phi_a\varphi^{-1}\right)\\ &= \left(\varphi(a),\varphi\phi_a\varphi^{-1}\right),\end{aligned}$$

by (4.5) in previous lemma. If we set $\bar{\phi}_{\varphi(a)} := \varphi\phi_a\varphi^{-1}$, for every $a \in B$, then

$$\begin{aligned}\bar{\phi}_{\varphi(a)}\bar{\phi}_{\varphi(b)} = \varphi\phi_a\varphi^{-1}\varphi\phi_b\varphi^{-1} = \varphi\phi_a\phi_b\varphi^{-1} = \varphi\phi_{a+\phi_a(b)}\varphi^{-1} = \bar{\phi}_{\varphi(a+\phi_a(b))}\\ = \bar{\phi}_{\varphi(a)+\varphi(\phi_a(b))} = \bar{\phi}_{\varphi(a)+\varphi(\phi_a(\varphi^{-1}(\varphi(b))))} = \bar{\phi}_{\varphi(a)+\bar{\phi}_a(\varphi(b))},\end{aligned}$$

for all $a, b \in B$. Hence, by previous lemma N_φ is a regular subgroup of $\mathrm{Hol}\,(B)$.

Given a right group B we establish a one-to-one correspondence between the class of left semi-braces with additive semigroup B and the set of all regular subgroups of the holomorph of B.

Theorem 4.3.5. *Let $(B,+)$ be a right group, $\mathcal{SB}$ the class of left semi-braces with additive semigroup $(B,+)$ and $\mathcal{R}$ the set of all regular subgroups of $\mathrm{Hol}\,(B)$ the holomorph of $(B,+)$.*

1. *If $B^\circ = (B,+,\circ) \in \mathcal{SB}$, then $N_{B^\circ} := \{(a,\lambda_a) \mid a \in B\} \in \mathcal{R}$.*

2. *The map $f : \mathcal{SB} \to \mathcal{R}, B^\circ \mapsto N_{B^\circ}$ is a bijection.*

Moreover, in this correspondence isomorphic left semi-braces correspond regular subgroups of $\mathrm{Hol}\,(B)$ conjugated under the action of $\mathrm{Aut}\,(B)$ and vice versa.

Proof. 1. Let B° be a left semi-brace. First, note that by Proposition 1.2.1 $\lambda_a \in \mathrm{Aut}\,(B)$, for every $a \in B$. The map $\tau : B \to \mathrm{Hol}\,(B)$ such that $\tau(a) = (a,\lambda_a)$ is a group homomorphism from the multiplicative group $(B,\circ)$ of the left semi-brace B° into the holomorph $\mathrm{Hol}\,(B)$. In fact, if $a, b \in B$, then

$$\begin{aligned}\tau(a)\,\tau(b) &= (a,\lambda_a)\,(b,\lambda_b) = (a+\lambda_a(b),\lambda_a\lambda_b) = (a+a\circ(a^- + b),\lambda_{a\circ b})\\ &= (a\circ b,\lambda_{a\circ b}) = \tau(a\circ b)\end{aligned}$$

moreover $\lambda_a\lambda_b = \lambda_{a+\lambda_a(b)}$ Hence $N_{B^\circ} = \tau(B^\circ)$ is a regular subgroup of $\mathrm{Hol}\,(B)$, by Lemma 4.3.4.
2. Let N be a regular subgroup of $\mathrm{Hol}\,(B)$. Then, by Lemma 4.3.4, there exists

$\phi : B \to \mathrm{Hol}(B)$ such that $N = \{(a, \phi_a) \mid a \in B\}$ and $\phi_a\phi_b = \phi_{a+\phi_a(b)}$, where $\phi_a := \phi(a)$. For all $a, b \in B$ we define on B the following binary operation

$$a \circ b := a + \phi_a(b).$$

Clearly $(B, \circ)$ is a group isomorphic to N. In fact $\mathrm{pr}_1 : N \to B, (a, \phi_a) \mapsto a$ is an isomorphism. Moreover denoted with a^- the element of B such that (a^-, ϕ_{a^-}) is the inverse of (a, ϕ_a) in N, we have that

$$\begin{aligned} a \circ b + a \circ (a^- + c) &= a + \phi_a(b) + a + \phi_a(a^- + c) \\ &= a + \phi_a(b) + a + \phi_a(a^-) + \phi_a(c) = a + \phi_a(b) + \phi_a(c) \\ &= a + \phi_a(b + c) = a \circ (b + c), \end{aligned}$$

since $a + \phi_a(a^-)$ is an idempotent by (4.7) in Lemma 4.3.4. Hence $(B, +, \circ)$ is a left semi-brace and $N_{B^\circ} = N$. Further if $B^\circ = (B, +, \circ)$ and $B^* = (B, +, *)$ are left semi-braces such that $f(B^\circ) = f(B^*)$, then $\{(a, \lambda^\circ_a) \mid a \in B\} = N_{B^\circ} = N_{B^*} = \{(a, \lambda^*_a) \mid a \in B\}$ and so

$$a \circ b = a + \lambda^\circ_a(b) = a + \lambda^*_a(b) = a * b,$$

for all $a, b \in B$. Therefore f is a bijection.
Finally, if $B^\circ = (B, +, \circ), B^* = (B, +, *) \in \mathcal{SB}$ such that there exists $\varphi : B^\circ \to B^*$ left semi-brace isomorphism, then, in particular $\varphi \in \mathrm{Aut}(B)$. Moreover, set $\lambda^\circ_a(b) := a \circ (a^- + b)$ and $\lambda^*_a(b) := a * (a^{-*} + b)$ where a^{-*} is the inverse of a in $(B, *)$, then $(0, \varphi) N_{B^\circ} (0, \varphi^{-1}) = N_{B^*}$. In fact,

$$\begin{aligned} (0, \varphi)(a, \lambda^\circ_a)\left(0, \varphi^{-1}\right) &= (0 + \varphi(a), \varphi\lambda^\circ_a)\left(0, \varphi^{-1}\right) = \left(\varphi(a) + \varphi\lambda^\circ_a(0), \varphi\lambda^\circ_a\varphi^{-1}\right) \\ &= \left(\varphi(a + a \circ (a^- + 0)), \varphi\lambda^\circ_a\varphi^{-1}\right) = \left(\varphi(a), \varphi\lambda^\circ_a\varphi^{-1}\right), \end{aligned}$$

for every $a \in B$. Moreover

$$\begin{aligned} \varphi\lambda^\circ_a\varphi^{-1}(b) &= \varphi\left(a \circ \left(a^- + \varphi^{-1}(b)\right)\right) = \varphi(a) * \left(\varphi(a)^{-*} + b\right) \\ &= \lambda^*_{\varphi(a)}(b), \end{aligned}$$

for all $a, b \in B$, i.e., $\varphi\lambda^\circ_a\varphi^{-1} = \lambda^*_{\varphi(a)}$, for every $a \in B$. Hence we obtain that $(0, \varphi)(a, \lambda^\circ_a)(0, \varphi^{-1}) = \left(\varphi(a), \lambda^*_{\varphi(a)}\right)$.
Conversely, let $N_1 := \left\{\left(a, \phi^{(1)}_a\right) \middle| a \in B\right\}, N_2 := \left\{\left(a, \phi^{(2)}_a\right) \middle| a \in B\right\}$ be regular subgroups of $\mathrm{Hol}(B)$ and $\varphi \in \mathrm{Aut}(B)$ such that $(0, \varphi) N_1 (0, \varphi^{-1}) = N_2$, where 0 is such that $\left(0, \phi^{(1)}_0\right)$ is the identity of N_1. Set $a \circ b := a + \phi^{(1)}_a(b)$ and $a * b := a + \phi^{(2)}_a(b)$, then φ is an isomorphism from $(B, +, \circ)$ into $(B, +, *)$. In fact, since $(0, \varphi) N_1 (0, \varphi^{-1}) =$

N_2, we have that $\left\{ \left(\varphi(a), \varphi\phi_a^{(1)}\varphi^{-1} \right) \;\middle|\; a \in B \right\} = \left\{ \left(a, \phi_a^{(2)} \right) \;\middle|\; a \in B \right\}$, i.e., $\varphi\phi_a^{(1)} = \phi_{\varphi(a)}^{(2)}\varphi$, for every $a \in B$. Hence

$$\begin{aligned} \varphi(a \circ b) &= \varphi\left(a + \phi_a^{(1)}(b)\right) = \varphi(a) + \varphi\phi_a^{(1)}(b) = \varphi(a) + \phi_{\varphi(a)}^{(2)}(\varphi(b)) \\ &= \varphi(a) * \varphi(b), \end{aligned}$$

in this way the left semi-braces corresponding to N_1 and N_2 respectively are isomorphic. □

Note that if $(B, +)$ is, in particular, a group, then $\mathrm{Hol}(B)$ is a group. As corollary of previous theorem we have the following theorem, see [19].

Theorem 4.3.6. *Let $(B, +)$ be a group, $\mathcal{SB}$ be the class of skew left braces with additive group $(B, +)$ and $\mathcal{R}$ the set of all regular subgroups of $\mathrm{Hol}(B)$ the holomorph of $(B, +)$.*

1. *If $B^\circ = (B, +, \circ) \in \mathcal{SB}$, then $N_{B^\circ} := \{(a, \lambda_a) \mid a \in B\} \in \mathcal{R}$.*

2. *The map $f : \mathcal{SB} \to \mathcal{R}, B^\circ \mapsto N_{B^\circ}$ is a bijection.*

Moreover, in this correspondence isomorphic skew left braces correspond regular subgroups of $\mathrm{Hol}(B)$ conjugated under the action of $\mathrm{Aut}(B)$.

Proof. By Theorem 4.3.5, if $N := \{(a, \phi_a) | \, a \in B\}$ is a regular subgroup of $\mathrm{Hol}(B)$, then, setting $a \circ b := a + \phi_a(b)$, we have that $(B, +, \circ)$ is a left semi-brace with $(B, +)$ group and, by Corollary 1.2.8, $(B, +, \circ)$ is a skew left brace. On the other hand, if $(B, +, \circ)$ is a skew left brace then, by Corollary 1.2.8, B is a left semi-brace and by previous theorem $\{(a, \lambda_a) | \, a \in B\}$ is a regular subgroup of $\mathrm{Hol}(B)$. □

Moreover if $(B, +)$ is an abelian group, then the same result holds for left braces (cf. Corollary 4.1.1 in [26]).

Theorem 4.3.7. *Let $(B, +)$ be a abelian group, $\mathcal{LB}$ be the class of left braces with additive group $(B, +)$ and $\mathcal{R}$ the set of all regular subgroups of $\mathrm{Hol}(B)$ the holomorph of $(B, +)$.*

1. *If $B^\circ = (B, +, \circ) \in \mathcal{LB}$, then $N_{B^\circ} := \{(a, \lambda_a) \mid a \in B\} \in \mathcal{R}$.*

2. *The map $f : \mathcal{LB} \to \mathcal{R}, B^\circ \mapsto N_{B^\circ}$ is a bijection.*

Moreover, in this correspondence isomorphic left braces correspond regular subgroups of $\mathrm{Hol}(B)$ conjugated under the action of $\mathrm{Aut}(B)$.

Finally, if we are interested on left F-braces we need to focus on the affine group $\mathrm{AGL}(V)$ of the vector space $(V,+)$ over F.

Definition 4.3.8. Let V be a vector space over a field F. The *affine group* of the vector space V is the group $\mathrm{AGL}(V) := V \times \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the linear group of V and the product is given by

$$(a,\alpha)(b,\beta) := (a+\alpha(b),\ \alpha\beta),$$

for all $a,b \in V$ and $\alpha,\beta \in \mathrm{GL}(V)$.

Remark 4.3.9. Let V be a vector space over a field F and $\mathrm{T}(V)$ the translation group of V. Then $\mathrm{AGL}(V)$ is the semidirect product of $\mathrm{T}(V)$ by $\mathrm{GL}(V)$.

Theorem 4.3.10 (Theorem 1. in [7])**.** *Let V be a vector space over a field F, $\mathcal{FB}$ be the class of left F-braces with underlying vector space V and $\mathcal{R}$ the set of all regular subgroups of $\mathrm{AGL}(V)$ the affine group of V.*

1. *If $V^\circ = (V,+,\circ) \in \mathcal{FB}$, then $N_{V^\circ} := \{(a,\lambda_a) \mid a \in V\} \in \mathcal{R}$.*

2. *The map $f : \mathcal{FB} \to \mathcal{R}, V^\circ \mapsto N_{V^\circ}$ is a bijection.*

Moreover, in this correspondence isomorphic left F-braces correspond regular subgroups of $\mathrm{AGL}(V)$ conjugated under the action of $\mathrm{GL}(V)$.

Proof. Clearly if $\alpha \in \mathrm{GL}(V)$, then $\alpha \in \mathrm{Aut}(V)$, where V is reviewed only as a group. Then $\mathrm{AGL}(V)$ is a subgroup of $\mathrm{Hol}(V)$. Therefore, if $V^\circ = (V,+,\circ)$ is a left F-brace, then in particular it is a left brace and so, by Theorem 4.3.7, $N_{V^\circ} := \{(a,\lambda_a) | a \in V\}$ is a regular subgroup of $\mathrm{Hol}(V)$. Moreover if $a,b \in V$ and $\mu \in F$, then

$$\begin{aligned}\mu\lambda_a(b) &= \mu(-a+a\circ b) = -\mu a + \mu(a\circ b) = -\mu a + (\mu-1)a + a\circ(\mu b)\\ &= -a + a\circ(\mu b) = \lambda_a(\mu b),\end{aligned}$$

i.e., $\lambda_a \in \mathrm{GL}(V)$. Hence N_{V° is a regular subgroup of $\mathrm{AGL}(V)$. On the other hand if $N := \{(a,\phi_a) | a \in V\}$ is a regular subgroup of $\mathrm{AGL}(V)$, then, by Theorem 4.3.7, setting $a\circ b := a + \phi_a(b)$, we have that $(V,+,\circ)$ is a left brace. Moreover if $a,b \in V$ and $\mu \in F$, then

$$\mu(a\circ b) = \mu(a+\phi_a(b)) = \mu a + \phi_a(\mu b) = a\circ(\mu b) + (\mu-1)a,$$

and so V is a left F-brace. □

Corollary 4.3.11 (Corollary 2. in [7]). *Let F be a field, $V^\circ = (V, +, \circ)$ a left F-brace, $\mathrm{T}(V)$ the translation group of V and $N_{V^\circ} = f(V^\circ)$ the regular subgroup of the affine group $\mathrm{AGL}(V)$ associated to V°. Then*

$$\mathrm{T}(V) \cap N_{V^\circ} = \{(a, \mathrm{id}_V) \mid a \in V,\ \forall b\ \in V \quad a + b = a \circ b\}.$$

Proof. Note that $T := \{(a, \mathrm{id}_V) \mid a \in V\}$. Then $a \in T \cap N_{V^\circ}$ if and only if $\lambda_a = \mathrm{id}_B$, i.e., $a \circ b = a + b$, for every $b \in V$. □

4.3.1 Regular subgroups of a 2-dimensional affine group

By Theorem 1.3.38 we may obtain all two-dimensional F-braces with non-trivial annihilator and then by Theorem 4.3.10 all regular subgroup of the affine group of dimension two with non-trivial intersection with the translations group.

Let N be the zero algebra of dimension one over a field F with basis (e_1) by Proposition 1.3.35 and Example 1.3.34, there exists θ a 2-cocycle of N if and only if there exists τ an automorphism of $(F, +)$.
Hence if F is a field all regular subgroups of $\mathrm{AGL}(F^2)$ with non trivial intersection with the translation group are given by

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & \tau(x) \\ 0 & 0 & 1 \end{pmatrix} \middle| \ x, y \in F \right\},$$

for every τ automorphism of $(F, +)$.
Here we have used the embedding of $\mathrm{AGL}(F^n)$ into $\mathrm{GL}(F, n+1)$ where we have that $\mathrm{AGL}(F^n)$ acts on the right on the set of affine vector $\Omega := \{(1, v) \mid v \in F^n\}$. Then, if N is a regular subgroup of $\mathrm{AGL}(F^n)$, there exists a unique element of N such that the embedding in $\mathrm{GL}(F, n+1)$ has $(1, v)$, as first row, for every $v \in F^n$, i.e.,

$$N = \left\{ \begin{pmatrix} 1 & v \\ 0 & \lambda_v \end{pmatrix} \middle| \ v \in F^n \right\}. \tag{4.8}$$

By Theorem 4.3.10 we note that there is a unique left F-brace V° such that $N = N_{V^\circ}$. Moreover the multiplication on the F-space V° is given by

$$x \circ y = x + \lambda_x(y),$$

for all $x, y \in V$. Conversely, if V° is a left F-brace and $\lambda_x : V \to V, y \mapsto x \circ (x^- + y)$, then the regular subgroup N_{V° has the description given in (4.8).

4.3.2 On Hegedűs' subgroups

In [22] Hegedűs constructed examples of regular subgroups of a specific affine group having trivial intersection with the translation group. In particular, let p be a prime and assume n is odd and $n \geq 3$ if $p = 2$, and $n \geq 4$ when p is odd. Let $\mathfrak{q}$ be a non-degenerate quadratic form on $\mathbb{F}_p^{n-1}$ and X the matrix of $\mathfrak{b}$ the polar form of $\mathfrak{q}$. If A is a matrix of order p in $O\left(\mathbb{F}_p^{n-1}, \mathfrak{q}\right)$ the orthogonal group associated with $\mathbb{F}_p^{n-1}$ and $\mathfrak{q}$, then

$$H := \left\{ \left(\begin{array}{c|cc} 1 & \mathfrak{q}(w)+k & w \\ \hline 0 & 1 & \mathbf{0} \\ \mathbf{0}^T & A^k X w^T & A^k \end{array} \right) \middle| \quad w \in \mathbb{F}_p^{n-1}, k \in \mathbb{F}_p \right\} \tag{4.9}$$

is a regular subgroup of $\mathrm{AGL}\left(\mathbb{F}_p^n\right)$ containing only the trivial translation.

Using the asymmetric product of left F-braces in the particular case of zero radical F-braces, we can extend Hegedűs' result.

For this purpose, let us note that if H, N are zero left F-braces we may simplify the condition (2.4) in Corollary 2.4.1 as in the following result.

Corollary 4.3.12. *Let F be a field of characteristic $p \neq 2$, and H, N zero left F-braces. A bilinear and symmetric map, $\mathfrak{b} : H \times H \to N$, and a homomorphism of the multiplicative group of N into the group of automorphisms of the left F-brace H, $\beta : N \to \mathrm{Aut}(H)$, satisfy* (2.4) *in Corollary 2.4.1 if and only if they satisfy the condition*

$$\mathfrak{b}(h_1, h_2) = ({}^n h_1, {}^n h_2) \tag{4.10}$$

for all $h_1, h_2 \in H$ and $n \in N$.

Proof. We prove that (2.4) in Corollary 2.4.1 is equivalent to (4.10), in the case of zero left F-braces. Suppose that (2.4) holds then

$$\begin{aligned} \mathfrak{b}(h_1, h_2) =& \mathfrak{b}(h_3 + {}^n h_1, h_3 + {}^n h_2) - \mathfrak{b}(h_3 + {}^n(h_1 + h_2), h_3) \\ =& \mathfrak{b}(h_3, h_3) + \mathfrak{b}(h_3, {}^n h_2) + \mathfrak{b}({}^n h_1, h_3) + \mathfrak{b}({}^n h_1, {}^n h_2) \\ & - \mathfrak{b}(h_3, h_3) - \mathfrak{b}({}^n h_1, h_3) - \mathfrak{b}({}^n h_2, h_3) \\ =& \mathfrak{b}({}^n h_1, {}^n h_2), \end{aligned}$$

for all $h_1, h_2, h_3 \in H$. Conversely suppose that (4.10), then, since H and N are zero left F-braces,

$$\begin{aligned}
&n \circ \mathfrak{b}(h_2, h_3) + \mathfrak{b}(h_1 \circ {}^{n}(h_2 + h_3), h_1) \\
&= n + \mathfrak{b}(h_2, h_3) + \mathfrak{b}(h_1 + {}^{n}h_2 + {}^{n}h_3, h_1) \\
&= n + \mathfrak{b}({}^{n}h_2, {}^{n}h_3) + \mathfrak{b}(h_1, h_1) + \mathfrak{b}({}^{n}h_2, h_1) + \mathfrak{b}({}^{n}h_3, h_1) \\
&= n + \mathfrak{b}(h_1 + {}^{n}h_2, h_1 + {}^{n}h_3) = n + \mathfrak{b}(h_1 \circ {}^{n}h_2, h_1 \circ {}^{n}h_3)
\end{aligned}$$

for all $h_1, h_2, h_3 \in H$ and $n \in N$. □

Corollary 4.3.13. *Let F be a field of characteristic 2, H a zero left F-brace and N the zero left F-brace with additive group $(F, +)$. A quadratic form, $\mathfrak{q}$, its polar form, $\mathfrak{b}$, and a homomorphism of the multiplicative group of N into the group of automorphism of the left F-brace H, $\beta : N \to \mathrm{Aut}(H)$, satisfy* (2.6) *in Corollary 2.4.4 if and only it they satisfy the condition*

$$\mathfrak{q}({}^{n}h) = \mathfrak{q}(h) \tag{4.11}$$

for all $h \in H$ and $n \in N$.

Proof. We prove that (2.6) in Corollary 2.4.4 is equivalent to (4.11). Suppose that (2.6) holds then

$$\mathfrak{q}({}^{n}h) = \mathfrak{b}({}^{n}h, 0) + \mathfrak{q}(h) + \mathfrak{q}(0) = \mathfrak{q}(h),$$

for every $h \in H$. On the other hand, suppose that (4.11) holds, then, since H and N are zero left F-braces,

$$\begin{aligned}
&n \circ \mathfrak{q}(h_2) + n + \mathfrak{q}(h_1) + \mathfrak{q}(h_1 \circ {}^{n}h_2) = \mathfrak{q}(h_2) + \mathfrak{q}(h_1) + \mathfrak{q}(h_1 + {}^{n}h_2) \\
&= \mathfrak{q}({}^{n}h_2) + \mathfrak{q}(h_1) + \mathfrak{q}(h_1 + {}^{n}h_2) = \mathfrak{b}(h_1 + {}^{n}h_2, h_1) = \mathfrak{b}(h_1 \circ {}^{n}h_2, h_1),
\end{aligned}$$

for all $h_1, h_2 \in H$ and $n \in N$. □

Remark 4.3.14. By Corollary 4.3.11 and Definition 1.3.15, we may check the intersection of the group associated to a left F-brace V with the translation group $\mathrm{T}(V)$ looking at the socle, i.e.,

$$N_V \cap \mathrm{T}(V) = \{(a, \mathrm{id}_V) \mid a \in \mathrm{Soc}(V)\}.$$

Hence, in assumptions of Corollary 4.3.12 we have that $(h, n) \in \mathrm{Soc}(H \rtimes_{\circ} N)$ if and only if $h \in \mathrm{rad}\,\mathfrak{b}$ and $\beta(n) = id_H$, where $\mathrm{rad}\,\mathfrak{b} = \{h \mid h \in H, \forall k \in H\ \mathfrak{b}(h, k) = 0\}$. In fact, $(h, n) \in \mathrm{Soc}(H \rtimes_{\circ} N)$, if and only for all $k \in H$ and $m \in N$ we have that $h + k = h + {}^{n}k$ and $\mathfrak{b}(h, k) + n + m = n + m$, i.e., $\beta(n) = \mathrm{id}_H$ and $h \in \mathrm{rad}\,\mathfrak{b}$.
In the same way, in the assumptions of Corollary 4.3.13 we have that $(h, n) \in \mathrm{Soc}(H \rtimes_{\circ} N)$ if and only if $h \in \mathrm{rad}\,\mathfrak{b}$ and $\beta(n) = \mathrm{id}_H$.

Theorem 4.3.15. *Let p be a prime and $m \in \mathbb{N}$. If one of the following conditions hold:*

1. *p odd, $m = 1$ and $n \geq 3$;*
2. *p odd, $m > 1$ and $n \geq 4$;*
3. *$p = 2$, $m = 1$ and $n = 3$ or $n \geq 5$;*
4. *$p = 2$, $m > 1$ and $n = 4, n = 5, n = 6$ or $n \geq 8$,*

then the affine group $\mathrm{AGL}(n+1, p^m)$ contains a regular subgroup having trivial intersection with the translation group.

Proof. If $m = 1$, $n \geq 3$ and p odd or $n \geq 2$, n even and $p = 2$, then set $H := \mathbb{F}_p^n$, where $\mathbb{F}_p$ is the field with p elements. There exists a quadratic form $\mathfrak{q} : V \to \mathbb{F}_p$, such that its polar form $\mathfrak{b}$ is non-degenerate. Moreover, since p divides the order of the orthogonal group $O(V, \mathfrak{q})$, there exists an element $A \in O(V, \mathfrak{q})$ of order p. Let $\beta : \mathbb{F}_p \to \mathrm{GL}(H)$ be the group homomorphism such that $\beta(1) = A$. Then, if p is odd, $H, \mathbb{F}_p, \beta, \mathfrak{b}$ satisfy the assumptions in Corollary 4.3.12, if $p = 2$, $H, \mathbb{F}_2, \beta, \mathfrak{q}$ satisfy the assumptions in Corollary 4.3.13. Hence for both cases, we may consider the asymmetric product of H by $\mathbb{F}_p$, i.e., $H \rtimes_\circ \mathbb{F}_p$ and by Theorem 4.3.10, the multiplicative group of this left $\mathbb{F}_p$-brace is a regular subgroup of the affine group $\mathrm{AGL}(n+1, p)$. Further, since β is faithful and $\mathfrak{b}$ is non-degenerate, by previous remark, its intersection with the translation group is trivial.

Now, suppose $m > 1$. If $n \geq 4$ and p is odd or $n \geq 4$, n is even and $p = 2$, set $H := \mathbb{F}_p^n$, then there exists an isotropic quadratic form $\mathfrak{q} : H \to \mathbb{F}_{p^m}$ such that the polar form $\mathfrak{b}$ is non-degenerate. Let h be an isotropic vector of H and $k_1 \in H$ such that $\mathfrak{b}(h, k_1) = 0$. Then we can choose $k_2, \ldots, k_m \in H$ such that $\mathfrak{b}(h, k_i) = 0$ and k_i does not lie on the planes generated by h and k_j with $j \leq i - 1$, for every $i \in \{2, \ldots, m\}$. Thus the matrices A_i associated to the Eichler trasformation (see 8.2.10 in [20])

$$\Sigma_{h,k_i}(x) := x + \mathfrak{b}(x, k_i)\, h - \mathfrak{b}(h, x)\, k_i - \mathfrak{b}(h, x)\, \mathfrak{q}(k_i)\, h,$$

for every $x \in H$, belong to the orthogonal group $O(V, \mathfrak{q})$, have order p and pairwise commute. Let

$$\beta : \mathbb{F}_{p^m} = \bigoplus_{i=1}^{m} \langle \omega_i \rangle \longrightarrow \mathrm{GL}(n, \mathbb{F}_{p^m})$$

be the group homomorphism such that $\beta(\omega_i) = A_i$. Obviously, β is injective, and if $h \in H$, $\lambda_1, \dots, \lambda_m \in \mathbb{F}_p$ then we have

$$\mathfrak{q}(h) = \mathfrak{q}\left(\prod_{i=1}^{m} A_i^{\lambda_i} h\right) = \mathfrak{q}\left(\sum_{i=1}^{m} \lambda_i \omega_i \, h\right),$$

since $\prod_{i=1}^{m} A_i^{\lambda_i} = \prod_{i=1}^{m} \beta(\omega_i)^{\lambda_i} = \beta\left(\sum_{i=1}^{m} \lambda_i \omega_i\right)$. Hence if $p = 2$ $H, \mathbb{F}_2, \mathfrak{q}, \beta$ satisfy the assumption of the Corollary 4.3.13. Moreover, if p is odd, by previous equality we have

$$\begin{aligned}
\mathfrak{b}(h_1, h_2) &= \mathfrak{q}(h_1 + h_2) - \mathfrak{q}(h_1) - \mathfrak{q}(h_2) \\
&= \mathfrak{q}\left(\sum_{i=1}^{m} \lambda_i \omega_i \, h_1 + \sum_{i=1}^{m} \lambda_i \omega_i \, h_2\right) - \mathfrak{q}\left(\sum_{i=1}^{m} \lambda_i \omega_i \, h_1\right) - \mathfrak{q}\left(\sum_{i=1}^{m} \lambda_i \omega_i \, h_2\right) \\
&= \mathfrak{b}\left(\sum_{i=1}^{m} \lambda_i \omega_i \, h_1, \sum_{i=1}^{m} \lambda_i \omega_i \, h_2\right),
\end{aligned}$$

for all $h_1, h_2 \in H$, $\lambda_1, \dots, \lambda_m \in \mathbb{F}_p$, then $H, \mathbb{F}_p, \mathfrak{b}, \beta$ satisfy the assumptions of the Corollary 4.3.12. Therefore the multiplicative group of the left $\mathbb{F}_{p^m}$-brace $\mathbb{F}_{p^m} \rtimes_\circ H$ is a regular subgroup of the affine group $\mathrm{AGL}(n+1, p^m)$. Further, since β is faithful and $\mathfrak{b}$ is non-degenerate, by previous remark, this subgroup has trivial intersection with the translation group.

For $p = 2$, if $m = 1$ and $n \geq 5$ or $m > 1$ $n \geq 9$, n odd, we may consider the direct product of two left $\mathbb{F}_{2^m}$-braces, $B_1 := \mathbb{F}_{2^m}^{n_1} \rtimes_\circ \mathbb{F}_{2^m}$ and $B_2 := \mathbb{F}_{2^m}^{n_2} \rtimes_\circ \mathbb{F}_{2^m}$ where n_1, n_2 are even $n_1, n_2 \geq 4$ such that $n_1 + 1 + n_2 + 1 = n + 1$. Therefore the multiplicative group of the $\mathbb{F}_{2^m}$-brace direct product is the direct product of the multiplicative groups of B_1 and B_2. Since $\mathrm{Soc}(B_1 \times B_2) = \mathrm{Soc}(B_1) \times \mathrm{Soc}(B_2)$ by Corollary 4.3.11 the intersection of the multiplicative group of $B_1 \times B_2$ with the translation group is trivial.

Finally, if $m \geq 1$, $p = 2$ and $n = 5$, there exists a regular subgroup of $\mathrm{AGL}(6, 2^m)$ by Example 3.4 in [27] that has trivial intersection with the translation group. □

Bibliography

[1]D. Bachiller. “Classification of braces of order p^3”. In: *J. Pure Appl. Algebra* 219.8 (2015), pp. 3568–3603.

[2]D. Bachiller. “Extensions, matched products, and simple braces”. In: *arXiv preprint arXiv:1511.08477* (2015).

[3]D. Bachiller and F. Cedó. “A family of solutions of the Yang-Baxter equation”. In: *J. Algebra* 412 (2014), pp. 218–229.

[4]D. Bachiller, F. Cedó, E. Jespers, and J. Okninski. “Iterated matched products of finite braces and simplicity; new solutions of the Yang-Baxter equation”. In: *arXiv preprint arXiv:1610.00477* (2016).

[5]R. Baxter. “Eight-vertex model in lattice statistics and one-dimensional anisotropic heisenberg chain. I. Some fundamental eigenvectors”. In: *Annals of Physics* 76.1 (1973), pp. 1 –24.

[6]A. Caranti, F. Dalla Volta, and M. Sala. “Abelian regular subgroups of the affine group and radical rings”. In: *Publ. Math. Debrecen* 69.3 (2006), pp. 297–308.

[7]F. Catino and R. Rizzo. “Regular subgroups of the affine group and radical circle algebras”. In: *Bull. Aust. Math. Soc.* 79.1 (2009), pp. 103–107.

[8]F. Catino, I. Colazzo, and P. Stefanelli. “Linear semi-cycle sets and bijective 1-cocycles”. In: *in preparation* (2017).

[9]F. Catino, I. Colazzo, and P. Stefanelli. “On regular subgroups of the affine group”. In: *Bull. Aust. Math. Soc.* 91.1 (2015), pp. 76–85.

[10]F. Catino, I. Colazzo, and P. Stefanelli. “Regular subgroups of the affine group and asymmetric product of radical braces”. In: *J. Algebra* 455 (2016), pp. 164–182.

[11]F. Catino, I. Colazzo, and P. Stefanelli. “Semi-braces and the Yang-Baxter equation”. In: *J. Algebra* 483 (2017), pp. 163–187.

[12]F. Catino, I. Colazzo, and P. Stefanelli. “The matched product of semi-braces and the Yang-Baxter equation”. In: *in preparation* (2017).

[13]F. Cedó, E. Jespers, and J. Okniński. “Braces and the Yang-Baxter equation”. In: *Comm. Math. Phys.* 327.1 (2014), pp. 101–116.

[14]A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. I*. Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1961, pp. xv+224.

[15]V. G. Drinfel′d. “On some unsolved problems in quantum group theory”. In: *Quantum groups (Leningrad, 1990)*. Vol. 1510. Lecture Notes in Math. Springer, Berlin, 1992, pp. 1–8.

[16]P. Etingof, T. Schedler, and A. Soloviev. “Set-theoretical solutions to the quantum Yang-Baxter equation”. In: *Duke Math. J.* 100.2 (1999), pp. 169–209.

[17]T. Gateva-Ivanova and M. Van den Bergh. “Semigroups of I-type”. In: *J. Algebra* 206.1 (1998), pp. 97–112.

[18]Pei Gu. “Another solution of Yang-Baxter equation on set and “metahomomorphisms on groups””. In: *Chinese Sci. Bull.* 42.22 (1997), pp. 1852–1855.

[19]L. Guarnieri and L. Vendramin. “Skew braces and the Yang-Baxter equation”. In: *Accepted for publication in Math. Comp. arXiv:1511.03171* (2015).

[20]A. J. Hahn and O. T. O’Meara. *The classical groups and K-theory*. Vol. 291. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With a foreword by J. Dieudonné. Springer-Verlag, Berlin, 1989, pp. xvi+576.

[21]M. Hall Jr. *The theory of groups*. The Macmillan Co., New York, N.Y., 1959, pp. xiii+434.

[22]P. Hegedűs. “Regular subgroups of the affine group”. In: *J. Algebra* 225.2 (2000), pp. 740–742.

[23]V. Lebed. “Cohomology of idempotent braidings, with applications to factorizable monoids”. In: *arXiv preprint arXiv:1607.08081* (2016).

[24]M. W. Liebeck, C. E. Praeger, and J. Saxl. “Regular subgroups of primitive permutation groups”. In: *Mem. Amer. Math. Soc.* 203.952 (2010), pp. vi+74.

[25]J.-H. Lu, M. Yan, and Y.-C. Zhu. “On the set-theoretical Yang-Baxter equation”. In: *Duke Math. J.* 104.1 (2000), pp. 1–18.

[26]D. K. Matsumoto. “Dynamical braces and dynamical Yang-Baxter maps”. In: *J. Pure Appl. Algebra* 217.2 (2013), pp. 195–206.

[27]M.A. Pellegrini and M.C. Tamburini Bellani. “Regular subgroups of the affine group with no translations”. In: *Pre-print https://arxiv.org/abs/1608.02492* (2017).

[28]W. Rump. “Braces, radical rings, and the quantum Yang-Baxter equation”. In: *J. Algebra* 307.1 (2007), pp. 153–170.

[29]W. Rump. “Modules over braces”. In: *Algebra Discrete Math.* 2 (2006), pp. 127–137.

[30]W. Rump. “Semidirect products in algebraic logic and solutions of the quantum Yang-Baxter equation”. In: *J. Algebra Appl.* 7.4 (2008), pp. 471–490.

[31]W. Rump. “The brace of a classical group”. In: *Note Mat.* 34.1 (2014), pp. 115–144.

[32]A. Soloviev. “Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation”. In: *Math. Res. Lett.* 7.5-6 (2000), pp. 577–596.

[33]C.N. Yang. “Some exact results for the many-body problem in one dimension with repulsive delta-function interaction”. In: *Physical Review Letters* 19.23 (1967), p. 1312.

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