The Asymmetric Product, a new construction of radical F-braces

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Aim

Finding all regular subgroups of an affine group is an open problem formalized by Liebeck, Praeger and Saxl in 2010 (see (5)). We introduce the asymmetric product of radical braces (see (1)), a construction which extends the semidirect product of radical braces and allows us to obtain rather systematic constructions of regular subgroups of the affine group. In particular, our approach puts in a more general context the regular subgroups of some affine groups constructed by Hegedűs in (4) that have trivial intersection with translation group.

Radical F-braces

Definition

A **(radical) brace** is a set B with two operation $+$ and $\circ$ such that $(B,+)$ is an abelian group, $(B,\circ)$ is a group and

$$(a+b)\circ c + c = a\circ c + b\circ c$$

for all $a,b,c\in B$.

Following Rump's terminology, a vector space V over a field F with a multiplication $\circ$ is called **(radical) F-brace** if V is a radical brace and

$$\lambda(u\circ v) = (\lambda u)\circ v + (\lambda - 1)v$$

holds for all $\lambda\in F$ and for all $u,v\in V$.

Clearly any associative algebra is an F-brace and any commutative F-brace is an associative algebra.

This is an important class of braces because there is a link between radical F-braces and regular subgroups of the affine group over the same vector space (Catino and Rizzo (2)).

Main Results

Theorem

Let S and T be radical braces, let $\mathfrak{b}: T\times T\to S$ be a symmetric cocycle on $(T,+)$ with values in $(S,+)$, and suppose there exists an action of the group $(S,\circ)$ on the radical brace T such that

$$\mathfrak{b}(t_1,t_2)\circ s + \mathfrak{b}((t_1+t_2)^s\circ t_3, t_3) = \mathfrak{b}(t_1^s\circ t_3, t_2^s\circ t_3) + s, \quad (1)$$

for all $s\in S$ and $t_1,t_2,t_3\in T$.
Then, the addition and the multiplication over $S\times T$ given by

$$(s_1,t_1)+(s_2,t_2) = (s_1+s_2+\mathfrak{b}(t_1,t_2),\ t_1+t_2)$$

$$(s_1,t_1)\circ(s_2,t_2) = (s_1\circ s_2,\ t_1^{s_2}\circ t_2)$$

define a structure of radical brace on $S\times T$.
We call this radical brace the **Asymmetric Product** of T by S (via $\mathfrak{b}$ and α) and denote it by $S\ltimes_\circ T$.

If $\mathfrak{b}$ is the null cocycle, then $S\ltimes_\circ T$ is the semidirect product of T by S (see (6) and also (3)).

Remark: The asymmetric product of two F-braces is not in general an F-brace.

Theorem

Let F be a field of characteristic $p\neq 2$, and let S and T be radical F-braces. Let $\mathfrak{b}: T\times T\to S$ be a bilinear and symmetric map and suppose there exists an action of the group $(S,\circ)$ on the radical F-brace T that satisfy the condition (1) of the previous theorem. Then the asymmetric product $S\ltimes_\circ T$ is a radical F-brace with the scalar multiplication given by

$$\lambda(s,t) = \left(\lambda s + \frac{\lambda(\lambda-1)}{2}\mathfrak{b}(t,t), \lambda t\right)$$

for all $\lambda\in F$, $s\in S$ and $t\in T$.

In the case of characteristic 2, the bilinearity of $\mathfrak{b}$ is not sufficient to obtain an F-brace.

We have only a partial result that involves a bilinear map that is the polar form of a quadratic one.

Proposition

Let V be a finite dimensional radical brace over a field F of characteristic 2, $\mathfrak{q}$ a quadratic form on V, $\mathfrak{b}$ the polar form of $\mathfrak{q}$ and suppose there exists an action of the group $(F,\circ)$ on the radical F-brace V. If

$$\mathfrak{b}(v_1^s\circ v_2, v_2) = \mathfrak{q}(v_1)\circ s + s + \mathfrak{q}(v_2) + \mathfrak{q}(v_1^s\circ v_2)$$

for all $v_1,v_2\in V$ and $s\in F$, then $F\ltimes_\circ V$ is a radical F-brace with scalar multiplication given by

$$\lambda(s_1,v_1) = (\lambda s_1 + \lambda(\lambda+1)\mathfrak{q}(v_1), \lambda v_1),$$

for all $\lambda, s_1\in F$ and $v_1\in V$.

Application

Theorem

Let p be a prime and $m\in\mathbb{N}$. If one of the following conditions hold:
1. p odd, $m=1$, $n\geq 3$;
2. $p=2$, $m=1$, $n=2$ or $n\geq 4$;
3. $m>1$, $n\geq 4$,

then the affine group $AGL(n+1,p^m)$ contains a regular subgroup having trivial intersection with the translation group.

Remark: If F is a field and V is a n-dimensional vector space over F, then the regular subgroup of $AGL(n+1,F)$ determined by $F\ltimes_\circ V$ intersects trivially the translations if and only if $\mathfrak{b}: V\times V\to F$ is non-degenerate and the action of $(F,\circ)$ on the radical F-brace V is faithful.

For example, let $m=1$, $n\geq 3$ and p odd or $n\geq 2$, n even and $p=2$ or $m>1$, $n\geq 4$ and p odd or $n\geq 4$, n even and $p=2$ and set $V := F_{p^m}^n$. Thus we may consider

- $\mathfrak{q}: V\to F_{p^m}$ a quadratic form such that the polar form $\mathfrak{b}$ associated to $\mathfrak{q}$ is non-degenerate;
- $A_1,\dots,A_m\in O(V,\mathfrak{q})$ that have order p and pairwise commute;
- the faithful action of $F_{p^m} = \bigoplus_{i=1}^m\langle\omega_i\rangle$ on the radical F_{p^m}-brace V such that $v^{\omega_i} = vA_i$, for all $v\in V$ and $i\in\{1,\dots,m\}$.

Then, the multiplicative group G of the F_{p^m}-brace $F_{p^m}\ltimes_\circ V$ is a regular subgroup of $AGL(n+1,p^m)$ that has the form

$$G = \left\{ \begin{pmatrix} 1 & \sum_{i=1}^m \mu_i\omega_i + \mathfrak{q}(v_2) & v_2 \\ 0 & 1 & 0 \\ 0 & \prod_{i=1}^m A_i^{\mu_i} X v_2^T & \prod_{i=1}^m A_i^{\mu_i} \end{pmatrix} \;\middle|\; \left(\sum_{i=1}^m \mu_i\omega_i, v_2\right)\in F_{p^m}\times V \right\},$$

where X is the matrix associated to $\mathfrak{b}$ respect to a fixed basis. Then this description includes Hegedűs' result (4) for $m=1$.

References

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