NEW CONSTRUCTION OF RADICAL F-BRACES: THE HOCHSCHILD PRODUCT

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Aim

Radical F-braces play an important role in the study of the regular subgroups of an affine group, in fact Catino and Rizzo in (2) established a link between regular subgroups of the affine group and the radical braces over a field on the underlying vector space.
In order to partially answer to the question of determining regular subgroup of an affine group, we exhibit the constructions of radical F-braces called **Hochschild Product**. In this way we obtain all radical F-braces with non-trivial annihilator.

Application

Let N be the one-dimensional zero algebra over a field F with a basis (e_1). Then let $\tau \in Aut(F^+)$, the map

$$\theta : N \times N \to F,\ (x_1e_1, y_1e_1) \mapsto x_1\tau(y_1)$$

is a 2-cocycle of the F-brace N but, in general not of the F-algebra N (in particular θ is a 2-cocycle of the one-dimensional zero algebra if and only if τ is linear).
Moreover these maps are the unique 2-cocycles of a one-dimensional zero algebra.
Hence we obtain all two-dimensional F-braces with non-trivial annihilator by Hochschild product N_θ of N by F.
In this way we obtain the regular subgroups of $AGL(2, F)$ determined by Hochschild product N_θ are conjugate to the following:

$$G = \left\{ \begin{pmatrix} 1 & b & w \\ 0 & 1 & \tau(b) \\ 0 & 0 & 1 \end{pmatrix} \middle|\ b, w \in F,\ \tau \in \mathrm{Aut}(F^+) \right\}$$

These are some examples provided by M. C. Tamburini Bellani in (4) yet.

Preliminary Results

Definition

A vector space V over a field F with a multiplication $\cdot$ is called a **(right) F-brace** if the following properties hold:
1. $(u + v) \cdot w = u \cdot w + v \cdot w;$
2. $u \cdot (v + w + v \cdot w) = u \cdot v + u \cdot w + (u \cdot v) \cdot w;$
3. $\alpha(u \cdot v) = (\alpha u) \cdot v,$

for all $\alpha \in F$ and for all $u, v, w \in V$.

Clearly any associative algebra is an F-brace and any commutative F-brace is an associative algebra.

Like in an ordinary algebra, let us introduce the **circle operation** in an F-brace V defined by $u \circ v := u+v+u\cdot v$, for all $u, v \in V$. Then $(V, \circ)$ is a semigroup. In particular, if $(V, \circ)$ is a group, then we say that the F-brace V is **radical**.

We define the set of **right annihilator** of an F-brace V and that of **left annihilator** respectively as follows:

$$Ann_R(V) := \{x \mid x \in V,\ \forall\, v \in V,\ \forall\, \lambda \in F\ v \cdot (\lambda x) = 0\}.$$

and

$$Ann_L(V) := \{x \mid x \in V,\ \forall\, v \in V,\ x \cdot v = 0\}.$$

Note that the previous definitions cannot be symmetric, since in general, if $v, w \in V$ and $\lambda \in F$, then $v \cdot (\lambda w) \neq \lambda(v \cdot w)$.
The set $Ann(V) := Ann_L(V) \cap Ann_R(V)$ is called the **annihilator** of the F-brace V.

The main result of (2) establishes the following link between regular subgroups of the affine group $AGL(V)$ and F- brace structures with the underlying vector space V.

Theorem

Let V be a vector space over a field F. Denote by $\mathcal{RB}$ the class of radical F-braces with underlying vector space V and by $\mathcal{T}$ the set of all regular subgroups of the affine group $AGL(V)$.
1. If $V^\bullet$ is a radical F-braces with underlying vector space V, then $T(V^\bullet) = \{\tau_x | x \in V\}$ is a regular subgroup of the affine group $AGL(V)$, where $\tau_x : V \to V, y \mapsto y \circ x$.
2. The map
$$f : \mathcal{RB} \longrightarrow \mathcal{T}, \quad V^\bullet \longmapsto T(V^\bullet)$$
is a bijection. In this correspondence, isomorphism classes of F-braces correspond to conjugacy classes under the action of $GL(V)$ of regular subgroups of $AGL(V)$.

Main Result

We translate the concepts of **2-cocycles** and the **Hochschild product** from the context of associative algebras into that of F-braces.

Definition

Let A be an F-brace and V a vector space over a field F. A map $\theta : A \times A \to V$ with the properties:
1. $\theta(\lambda a + \mu b, c) = \lambda(\theta(a, c)) + \mu(\theta(b, c))$;
2. $\theta(a, b + c + b \cdot c) = \theta(a, b) + \theta(a, c) + \theta(a \cdot b, c)$,

for all $a, b, c \in A$ and $\lambda, \mu \in F$, is called a **2-cocycle** of A with values in V.

Thus 2-cocycles of F-algebras are particular cases of 2-cocycles of F-braces.
But if we regard an F-algebra as an F-brace, then a 2-cocycle in the sense of the previous definition is not necessarily a 2-cocycle in the usual sense.

Definition

Let A be an F-brace, V an F-vector space, $\theta : A \times A \to V$ a 2-cocycle. Put $A_\theta := A \oplus V$. For all $a, b \in A$ and $v, w \in V$ we define

$$(a + v) \cdot (b + w) := a \cdot b + \theta(a, b).$$

The F-brace A_θ is called a **Hochschild product** of A by V.

Remark: if A is a radical F-brace and θ is a 2-cocycle of A with values in an F-vector space V. Then A_θ is radical.

Theorem

Let B be a radical F-brace such that $Ann(B) \neq \{0\}$. Then there exist an F-brace A, an F-vector space V and a 2-cocycle $\theta : A \times A \to V$ such that B is isomorphic to A_θ.

Proof sketch. We consider $A := B/Ann(B)$, $V := Ann(B)$. If $\pi : B \to A$ be the projection map and we choose a linear map $\sigma : A \to B$ such that $\pi(x\sigma) = x$, for all $x \in A$, then we obtain a function θ from $A \times A$ into V by defining

$$\theta(x, y) := \sigma(x) \cdot \sigma(y) - \sigma(x \cdot y).$$

that is a 2-cocycle. So B is isomorphic to A_θ

References

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