Aim
Radical F-braces play an important role in the study of the regular subgroups of an affine group, in fact Catino and Rizzo in (2) established a link between regular subgroups of the affine group and the radical braces over a field on the underlying vector space. In order to partially answer to the question of determining regular subgroup of an affine group, we exhibit the constructions of radical F-braces called Hochschild Product. In this way we obtain all radical F-braces with non-trivial annihilator.

Preliminary Results
We define the set of right annihilator of an F-brace V and that of left annihilator respectively as follows:

\[ Ann_{R}(V) := \{ x \mid x \in V, \forall v \in V, \forall \lambda \in F, v \cdot (\lambda x) = 0 \}. \]

and

\[ Ann_{L}(V) := \{ x \mid x \in V, \forall v \in V, x \cdot v = 0 \}. \]

Note that the previous definitions cannot be symmetric, in general if \( v, w \in V \) and \( \lambda \in F \), then \( v \cdot (\lambda w) \neq \lambda (v \cdot w) \). The set \( Ann_{L}(V) \cap Ann_{R}(V) \) is called the annihilator of the F-brace V. The main result of (2) establishes the following link between regular subgroups of the affine group \( AGL(V) \) and F-brace structures with the underlying vector space V.

Application
Let \( N \) be the one-dimensional zero algebra over a field \( F \) with a basis \( \{ a \} \). Then let \( \tau \in \text{Aut}(F^*) \), the map

\[ \theta : N \times N \rightarrow F, \quad (x, y) \mapsto x, \tau(y) \]

is a 2-cocycle of the F-brace N but, in general not of the F-algebra N (in particular \( \theta \) is a 2-cocycle of the one-dimensional zero algebra if and only if \( \tau \) is linear).

Moreover these maps are the unique 2-cocycles of a one-dimensional zero algebra. Hence we obtain all two-dimensional F-braces with non-trivial annihilator by Hochschild product \( N \times \text{Hom}(N,F) \) of \( N \) by \( F \).

In this way we obtain the regular subgroups of \( AGL(2,F) \) determined by Hochschild product \( N \times \text{Hom}(N,F) \) are conjugate to the following:

\[ G = \begin{cases} \{ 1, b \} & b, w \in F, \; \tau \in \text{Aut}(F^*) \end{cases} \]

These are some examples provided by M. C. Tamburini Bellani in (4) yet.

Main Result
We translate the concepts of 2-cocycles and the Hochschild product from the context of associative algebras into that of F-braces.

Definition
Let A be an F-brace and V a vector space over a field F. A map \( \theta : A \times A \rightarrow V \) with the properties:

1. \( \theta(ab + bc) = \lambda(\theta(a,c) + \theta(b,c)) \);
2. \( \theta(a,b + c + b) = \theta(a,b) + \theta(a,c) + \theta(a,b,c) \),

for all \( a, b, c \in A \) and \( \lambda \in F \), is called a 2-cocycle of \( A \) with values in \( V \).

Thus 2-cocycles of F-braces are particular cases of 2-cocycles of F-braces. But if we regard an F-brace as an F-brace, then a 2-cocycle in the sense of the previous definition is not necessarily a 2-cocycle in the usual sense.

Definition
Let \( A \) be an F-brace, \( V \) an F-vector space, \( \theta : A \times A \rightarrow V \) a 2-cocycle. Put \( A_{\theta} = A \oplus V \). For all \( a, b \in A \) and \( x, y \in V \) we define

\[ (a + x) \cdot (b + y) = a \cdot b + \theta(a,b) + x \cdot y. \]

The F-brace \( A_{\theta} \) is called a Hochschild product of \( A \) by \( V \).

Remark: If \( A \) is a radical F-brace and \( \theta \) is a 2-cocycle of \( A \) with values in an F-vector space \( V \). Then \( A_{\theta} \) is radical.

References

Theorem
Let \( B \) be a radical F-brace such that \( Ann(B) \neq \{ 0 \} \). Then there exist an F-brace \( A \), an F-vector space \( V \) and a 2-cocycle \( \theta : A \times A \rightarrow V \) such that \( B \) is isomorphic to \( A_{\theta} \).

Proof sketch: We consider \( A = B/Ann(B) \), \( V = Ann(B) \).
If \( \pi : B \rightarrow A \) be the projection map and we choose a linear map \( \sigma : A \rightarrow B \) such that \( \pi(x) = x \), for all \( x \in A \), then we obtain a function \( \sigma \), from \( A \times A \) into \( B \) by defining

\[ \theta(x, y) = \sigma(x) \cdot \sigma(y) - \sigma(x \cdot y) \]

that is a 2-cocycle. So \( B \) is isomorphic to \( A_{\theta} \).