Skew braces with non-trivial annihilator

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Lecce, September 5th, 2017

The Yang-Baxter equation

Set-theoretical solutions of the Yang-Baxter equation (I)

In 2017, Guarnieri and Vendramin introduced skew brace in order to find set-theoretical solution of the Yang-Baxter equation.

If X is a set, a (set-theoretical) **solution of the Yang-Baxter equation** $r : X \times X \to X \times X$ is a map such that the well-known **braid equation**

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

is satisfied, where $r_1 = r \times \mathrm{id}_X$ and $r_2 = \mathrm{id}_X \times r$.

Problem

How to obtain and construct all solutions of the Yang-Baxter equation?

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In particular, if X is a set, $r : X \times X \to X \times X$ is a solution and $a, b \in X$, then we denote

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

where λ_a, ρ_b are maps from X into itself.

We say that r is

- **involutive** if $r^2 = \mathrm{id}_{X \times X}$;
- **left non-degenerate** if λ_a is bijective, for every $a \in X$;
- **right non-degenerate** if ρ_b is bijective, for every $b \in X$;
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Involutive solutions and braces

- In 1999, Etingof, Schedler, and Soloviev, and independently Gateva-Ivanova and Van den Bergh initially studied non-degenerate involutive solutions in terms of group theory.
- In 2007, Rump introduced a generalization of the notion of radical rings named **brace**.
- In 2014, Cedó, Jespers and Okniński provided an equivalent definition of braces in terms of groups.

Definition

Let B be a set with two operations $+$ and $\circ$ such that $(B,+)$ is an abelian group and $(B,\circ)$ is a group. We say that $(B,+,\circ)$ is a (left) brace if

$$a\circ(b+c)+a=a\circ b+a\circ c,$$

holds for all $a,b,c\in B$.

For instance, if $(R,+,\cdot)$ is a radical ring and if we consider the adjoint operation defined by $a\circ b:=a\cdot b+a+b$, for all $a,b\in R$, then $(R,+,\circ)$ is a brace.

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Skew brace

Solution

If $(B, +, \circ)$ is a skew brace, then the map $r : B \times B \to B \times B$ defined by

$$r(a, b) := \left(a \circ \left(a^{-} + b\right),\ \left(a^{-} + b\right)^{-} \circ b\right)$$

is a solution, called **solution associated to the skew brace** B.

It is possible to prove that if B is a skew brace and $r : B \times B \to B \times B$ is the solution associated to B, then r is bijective and both left and right non-degenerate.

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Examples

- Clearly, if $(B,+,\circ)$ is a brace than it is also a skew brace.
- If $(B,\circ)$ is a group and we define $a+b:=a\circ b$, we have that $(B,+,\circ)$ is a skew brace that we call **zero skew brace**. The solution associated to this skew brace is the map $r: B\times B\to B\times B$ defined by

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Moreover, note that if $(B,+,\circ)$ is a skew brace with additive structure an abelian group, then B is a brace.

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Ideal

If B is a skew brace, then we define the map $\lambda_a : B \to B$ by

$$\lambda_a(b) = a \circ \left(a^- + b\right),$$

for every $a \in B$. A normal subgroup of $(B, \circ)$ is said to be an **ideal** of B if $I + a = a + I$ and $\lambda_a(I) \subseteq I$, for every $a \in B$.

An important example of ideal is the **socle** defined by

$$\mathrm{Soc}(B) := \{a \mid a \in B,\ \forall b \in B \quad a + b = a \circ b, a + b = b + a\}.$$

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Annihilator

Definition (F. Catino, I.C., P. Stefanelli, in preparation)

If B is a skew brace then the **annihilator** is the set given by

$$\mathrm{Ann}\,(B) = \{a \mid a \in B \quad \forall\, b \in B \quad a \circ b = a + b = b + a = b \circ a\}\,.$$

Note that

$$\mathrm{Ann}\,(B) = \mathrm{Soc}\,(B) \cap \mathrm{Z}\,(B)\,,$$

where $\mathrm{Z}\,(B)$ is the centre of $(B, \circ)$, and that if $a \in \mathrm{Soc}\,(B)$, then $a^- = -a$. Hence, it is easy to prove that $\mathrm{Ann}\,(B)$ is normal subgroup of both $(B, \circ)$ and $(B, +)$. Moreover, if $a \in \mathrm{Ann}\,(B)$ and $b \in B$, then

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Therefore, $\mathrm{Ann}\,(B)$ is an ideal of B.

Skew brace

Annihilator

Definition (F. Catino, I.C., P. Stefanelli, in preparation)

If B is a skew brace then the **annihilator** is the set given by

$$\mathrm{Ann}(B) = \{a \mid a \in B \quad \forall\, b \in B \quad a \circ b = a + b = b + a = b \circ a\}\,.$$

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Describe all skew braces with non-trivial annihilator

Problem

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For this purpose, we recall some classical cohomological definition for groups that we use in the main theorem, that allow us to describe all skew braces with non-trivial annihilator.

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Tools

2-cocycles

If $(B,+)$ is a group and $(I,+)$ is an abelian group, then a map $\tau : B \times B \to I$ is a **2-cocycle from $(B,+)$ with values in $(I,+)$** if the following conditions hold:

1. $\tau(b_1 + b_2, b_3) + \tau(b_1, b_2) = \tau(b_1, b_2 + b_3) + \tau(b_2, b_3)$, for all $b_1, b_2, b_3 \in B$;
2. $\tau(b, 0) = \tau(0, b) = 0$, for every $b \in B$.

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Skew brace

Hochschild pair

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Let $(B, +, \circ)$ be a skew brace, $(I, +)$ an abelian group. A pair (τ, θ) such that $\tau : B \times B \to I$ is a 2-cocycle from $(B, +)$ with values in I, $\theta : B \times B \to A$ is a 2-cocycle of $(B, \circ)$ with values in I and they satisfy

$$\begin{aligned}\theta(b_1, b_2 + b_3) + \tau(b_2, b_3) = \; & \theta(b_1, b_2) + \theta(b_1, b_3) \\ & -\tau(b_1, -b_1 + b_1 \circ b_3) \\ & +\tau(b_1 \circ b_2, -b_1 + b_1 \circ b_3),\end{aligned}$$

for all $b_1, b_2, b_3 \in B$, is said to be a **Hochschild pair of the skew brace B with values in I**.

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Skew brace with non-trivial annihilator

Main theorem (I)

Proposition (F. Catino, I.C., P. Stefanelli, in preparation)

Let $(B, +, \circ)$ be a skew brace, $(I, +)$ an abelian group and (τ, θ) a Hochschild pair of B with values in I. If we define on the cartesian product $B \times I$

$$(b_1, i_1) + (b_2, i_2) := (b_1 + b_2, i_1 + i_2 + \tau(b_1, b_2)),$$
$$(b_1, i_1) \circ (b_2, i_2) := (b_1 \circ b_2,\ i_1 + i_2 + \theta(b_1, b_2)),$$

for all $b_1, b_2 \in B$, $i_1, i_2 \in I$, then $(B \times I, +, \circ)$ is a skew brace. We call this skew brace the **Hochschild product of B by I (via τ and θ)**.

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Skew braces with non-trivial annihilator

Main theorem (II)

Theorem (F. Catino, I.C., P. Stefanelli, in preparation)

Let B be a skew brace such that $\operatorname{Ann}(B) \neq 0$ and $I := \operatorname{Ann}(B)$. Then there exists a Hochschild pair (τ, θ) of the skew brace $\bar{B} := B/I$ with values in I such that B is isomorphic to the Hochschild product of $\bar{B}$ by I (via τ and θ).

Sketch of the proof.

- Consider $\pi : B \to \bar{B}$ the projection map and $s : \bar{B} \to B$ a map such that $s(\bar{0}) = 0$ and $\pi(s(\bar{b})) = \bar{b}$, for every $\bar{b} \in \bar{B}$.
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Braces with non-trivial annihilator

We can specialize this result to braces, i.e., skew braces with the additive structure an abelian group.

Recall that if $(B,+)$, $(I,+)$ are abelian groups, then a map $\tau : B \times B \to I$ is a **symmetric 2-cocycle from $(B,+)$ with values in $(I,+)$** if τ is a 2-cocycle and moreover the following condition holds:

3. $\tau(b_1, b_2) = \tau(b_2, b_1)$, for all $b_1, b_2 \in B$.

Corollary

Let $(B,+,\circ)$ be a brace, $(I,+)$ an abelian group, (τ,θ) a Hochschild pair of the brace B with values in I, with τ **symmetric**. Then the Hochschild product of B by I (via τ and θ) is a brace.

Corollary

Let B be a brace such that $\mathrm{Ann}(B) \neq 0$ and $I := \mathrm{Ann}(B)$. Then there exists a Hochschild pair (τ,θ) of the brace $\bar{B} := B/I$ with values in I, with τ **symmetric**, such that B is isomorphic to the Hochschild product of $\bar{B}$ by I (via τ and θ).

Braces with non-trivial annihilator

We can specialize this result to braces, i.e., skew braces with the additive structure an abelian group.

Recall that if $(B, +)$, $(I, +)$ are abelian groups, then a map $\tau : B \times B \to I$ is a **symmetric 2-cocycle from $(B, +)$ with values in $(I, +)$** if τ is a 2-cocycle and moreover the following condition holds:

3. $\tau(b_1, b_2) = \tau(b_2, b_1)$, for all $b_1, b_2 \in B$.

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Thanks for your attention!