



Skew braces with non-trivial annihilator

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The Yang-Baxter equation

Set-theoretical solutions of the Yang-Baxter equation (I)

In 2017, Guarnieri and Vendramin introduced skew brace in order to find set-theoretical solution of the Yang-Baxter equation.

If X is a set, a (set-theoretical) **solution of the Yang-Baxter equation** $r : X \times X \rightarrow X \times X$ is a map such that the well-known **braid equation**

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

is satisfied, where $r_1 = r \times \text{id}_X$ and $r_2 = \text{id}_X \times r$.

Problem

How to obtain and construct all solutions of the Yang-Baxter equation?

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Set-theoretical solutions of the Yang-Baxter equation (II)

In particular, if X is a set, $r : X \times X \rightarrow X \times X$ is a solution and $a, b \in X$, then we denote

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

where λ_a, ρ_b are maps from X into itself.

We say that r is

- ▶ **involution** if $r^2 = \text{id}_{X \times X}$;
- ▶ **left non-degenerate** if λ_a is bijective, for every $a \in X$;
- ▶ **right non-degenerate** if ρ_b is bijective, for every $b \in X$;
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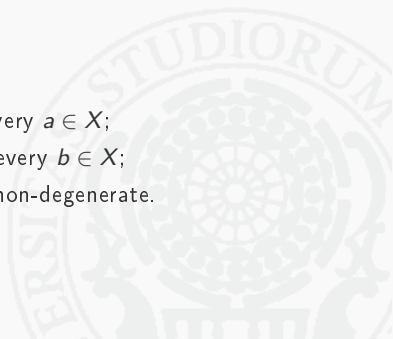
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Involutive solutions and braces

- ▶ In 1999, Etingof, Schedler, and Soloviev, and independently Gateva-Ivanova and Van den Bergh initially studied non-degenerate involutive solutions in terms of group theory.
- ▶ In 2007, Rump introduced a generalization of the notion of radical rings named **brace**.
- ▶ In 2014, Cedó, Jespers and Okniński provided an equivalent definition of braces in terms of groups.

Definition

Let B be a set with two operations $+$ and \circ such that $(B, +)$ is an abelian group and (B, \circ) is a group. We say that $(B, +, \circ)$ is a (left) brace if

$$a \circ (b + c) + a = a \circ b + a \circ c,$$

holds for all $a, b, c \in B$.

For instance, if $(R, +, \cdot)$ is a radical ring and if we consider the adjoint operation defined by $a \circ b := a \cdot b + a + b$, for all $a, b \in R$, then $(R, +, \circ)$ is a brace.

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Non-degenerate solutions and skew braces

- ▶ In 2000, **Lu, Yan** and **Zhu** and independently **Soloviev** initially studied non-degenerate bijective solutions not necessarily involutive.
- ▶ In 2017, Guarnieri and Vendramin introduced a generalization of the notion of braces named **skew brace**.

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Let B be a set with two operations $+$ and \circ such that $(B, +)$ and (B, \circ) are groups. We say that $(B, +, \circ)$ is a skew (left) brace if

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Skew brace

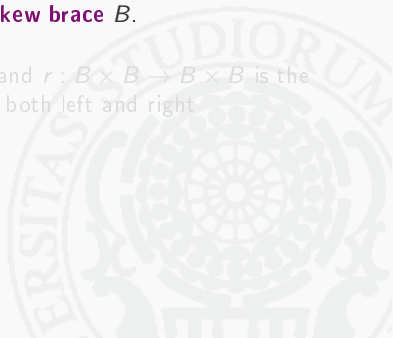
Solution

If $(B, +, \circ)$ is a skew brace, then the map $r : B \times B \rightarrow B \times B$ defined by

$$r(a, b) := (a \circ (a^- + b), (a^- + b)^- \circ b)$$

is a solution, called **solution associated to the skew brace B** .

It is possible to prove that if B is a skew brace and $r : B \times B \rightarrow B \times B$ is the solution associated to B , then r is bijective and both left and right non-degenerate.



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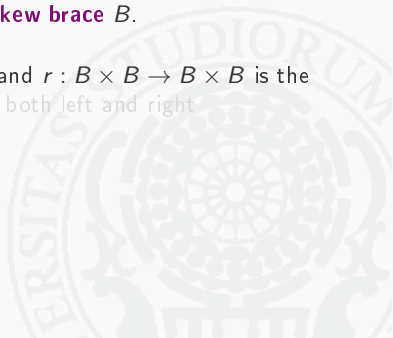
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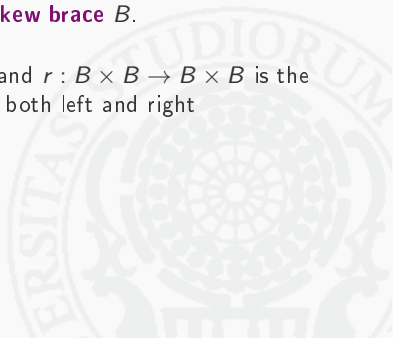
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Examples

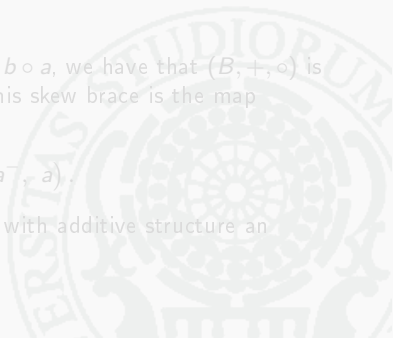
- ▶ Clearly, if $(B, +, \circ)$ is a brace than it is also a skew brace.
- ▶ If (B, \circ) is a group and we define $a + b := a \circ b$, we have that $(B, +, \circ)$ is a skew brace that we call **zero skew brace**. The solution associated to this skew brace is the map $r : B \times B \rightarrow B \times B$ defined by

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Moreover, note that if $(B, +, \circ)$ is a skew brace with additive structure an abelian group, then B is a brace.



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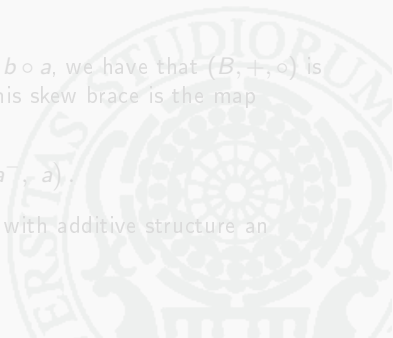
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Ideal

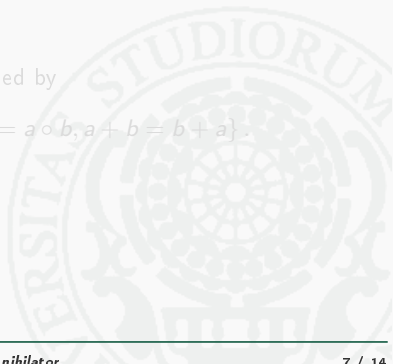
If B is a skew brace, then we define the map $\lambda_a : B \rightarrow B$ by

$$\lambda_a(b) = a \circ (a^- + b),$$

for every $a \in B$. A normal subgroup of (B, \circ) is said to be an **ideal** of B if $I + a = a + I$ and $\lambda_a(I) \subseteq I$, for every $a \in B$.

An important example of ideal is the **socle** defined by

$$\text{Soc}(B) := \{ a \mid a \in B, \forall b \in B \quad a + b = a \circ b, a + b = b + a \}.$$



Skew brace

Ideal

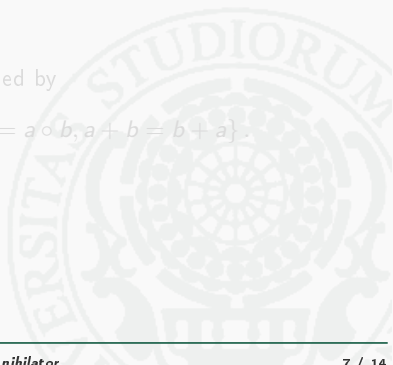
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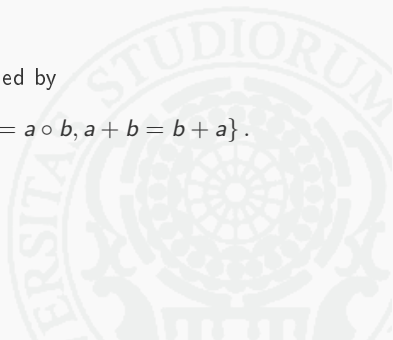
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Definition (F. Catino, I.C., P. Stefanelli, in preparation)

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where $Z(B)$ is the centre of (B, \circ) , and that if $a \in \text{Soc}(B)$, then $a^- = -a$. Hence, it is easy to prove that $\text{Ann}(B)$ is normal subgroup of both (B, \circ) and $(B, +)$. Moreover, if $a \in \text{Ann}(B)$ and $b \in B$, then

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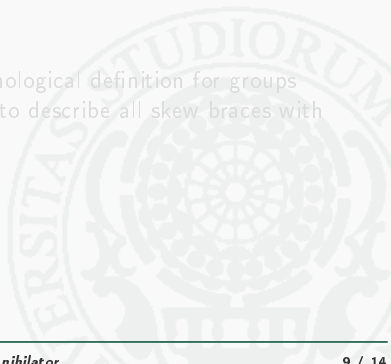
The aim

Describe all skew braces with non-trivial annihilator

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For this purpose, we recall some classical cohomological definition for groups that we use in the main theorem, that allow us to describe all skew braces with non-trivial annihilator.



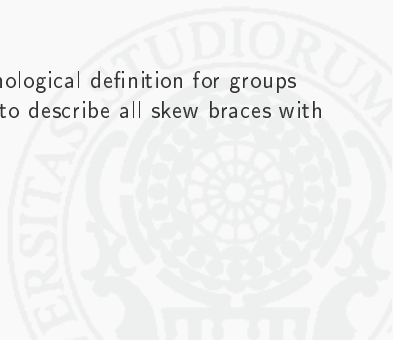
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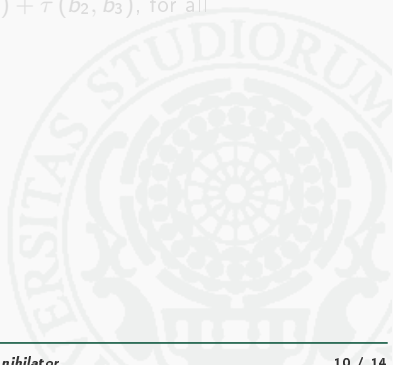


Tools

2-cocycles

If $(B, +)$ is a group and $(I, +)$ is an abelian group, then a map $\tau : B \times B \rightarrow I$ is a **2-cocycle from $(B, +)$ with values in $(I, +)$** if the following conditions hold:

1. $\tau(b_1 + b_2, b_3) + \tau(b_1, b_2) = \tau(b_1, b_2 + b_3) + \tau(b_2, b_3)$, for all $b_1, b_2, b_3 \in B$;
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Skew brace

Hochschild pair

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Let $(B, +, \circ)$ be a skew brace, $(I, +)$ an abelian group. A pair (τ, θ) such that $\tau : B \times B \rightarrow I$ is a 2-cocycle from $(B, +)$ with values in I , $\theta : B \times B \rightarrow A$ is a 2-cocycle of (B, \circ) with values in I and they satisfy

$$\begin{aligned} \theta(b_1, b_2 + b_3) + \tau(b_2, b_3) = & \theta(b_1, b_2) + \theta(b_1, b_3) \\ & - \tau(b_1, -b_1 + b_1 \circ b_3) \\ & + \tau(b_1 \circ b_2, -b_1 + b_1 \circ b_3), \end{aligned}$$

for all $b_1, b_2, b_3 \in B$, is said to be a **Hochschild pair of the skew brace B with values in I** .

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Skew brace with non-trivial annihilator

Main theorem (I)

Proposition (F. Catino, I.C., P. Stefanelli, in preparation)

Let $(B, +, \circ)$ be a skew brace, $(I, +)$ an abelian group and (τ, θ) a Hochschild pair of B with values in I . If we define on the cartesian product $B \times I$

$$(b_1, i_1) + (b_2, i_2) := (b_1 + b_2, i_1 + i_2 + \tau(b_1, b_2)),$$

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for all $b_1, b_2 \in B$, $i_1, i_2 \in I$, then $(B \times I, +, \circ)$ is a skew brace. We call this skew brace the **Hochschild product of B by I (via τ and θ)**.

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Skew braces with non-trivial annihilator

Main theorem (II)

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Let B be a skew brace such that $\text{Ann}(B) \neq 0$ and $I := \text{Ann}(B)$. Then there exists a Hochschild pair (τ, θ) of the skew brace $\bar{B} := B/I$ with values in I such that B is isomorphic to the Hochschild product of \bar{B} by I (via τ and θ).

Sketch of the proof.

- ▶ Consider $\pi : B \rightarrow \bar{B}$ the projection map and $s : \bar{B} \rightarrow B$ a map such that $s(\bar{0}) = 0$ and $\pi(s(\bar{b})) = \bar{b}$, for every $\bar{b} \in \bar{B}$.
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- ▶ Prove that $\tau : \bar{B} \times \bar{B} \rightarrow I$ defined by

$$\tau(\bar{b}_1, \bar{b}_2) = s(\bar{b}_1) + s(\bar{b}_2) - s(\bar{b}_1 + \bar{b}_2),$$

for all $\bar{b}_1, \bar{b}_2 \in \bar{B}$, is a 2-cocycle from $(\bar{B}, +)$ with values in $(I, +)$.

- ▶ Prove that $\theta : \bar{B} \times \bar{B} \rightarrow A$, defined by

$$\theta(\bar{b}_1, \bar{b}_2) = (s(\bar{b}_1 \circ \bar{b}_2))^- \circ s(\bar{b}_1) \circ s(\bar{b}_2),$$

for all $\bar{b}_1, \bar{b}_2 \in \bar{B}$, is a 2-cocycle from (\bar{B}, \circ) with values in $(I, +)$.

- ▶ Finally, (τ, θ) is a Hochschild pair and the Hochschild product of \bar{B} by I (via τ and θ) is isomorphic to B .

Skew braces with non-trivial annihilator

Main theorem (II)

Theorem (F. Catino, I.C., P. Stefanelli, in preparation)

Let B be a skew brace such that $\text{Ann}(B) \neq 0$ and $I := \text{Ann}(B)$. Then there exists a Hochschild pair (τ, θ) of the skew brace $\bar{B} := B/I$ with values in I such that B is isomorphic to the Hochschild product of \bar{B} by I (via τ and θ).

Sketch of the proof.

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Braces with non-trivial annihilator

We can specialize this result to braces, i.e., skew braces with the additive structure an abelian group.

Recall that if $(B, +)$, $(I, +)$ are abelian groups, then a map $\tau : B \times B \rightarrow I$ is a **symmetric 2-cocycle from $(B, +)$ with values in $(I, +)$** if τ is a 2-cocycle and moreover the following condition holds:

$$3. \tau(b_1, b_2) = \tau(b_2, b_1), \text{ for all } b_1, b_2 \in B.$$

Corollary

Let $(B, +, \circ)$ be a brace, $(I, +)$ an abelian group, (τ, θ) a Hochschild pair of the brace B with values in I , with τ **symmetric**. Then the Hochschild product of B by I (via τ and θ) is a brace.

Corollary

Let B be a brace such that $\text{Ann}(B) \neq 0$ and $I := \text{Ann}(B)$. Then there exists a Hochschild pair (τ, θ) of the brace $\bar{B} := B/I$ with values in I , with τ **symmetric**, such that B is isomorphic to the Hochschild product of \bar{B} by I (via τ and θ).

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Thanks for your attention!

