



Groups, Rings
and the Yang-Baxter
equation



Semi-braces and the Yang-Baxter equation

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Spa, June 22nd, 2017

The Yang-Baxter equation

If X is a set, a (set-theoretical) **solution** of the Yang-Baxter equation $r : X \times X \rightarrow X \times X$ is a map such that the well-known **braid equation**

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

is satisfied, where $r_1 = r \times \text{id}_X$ and $r_2 = \text{id}_X \times r$.

Problem

How to obtain and construct all solutions of the Yang-Baxter equation?

Determining all set-theoretic solutions of the Yang-Baxter equation is a very difficult task. Even if we may find several works about this topic, it is still an open problem.

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Solutions of the Yang-Baxter equation

In particular, if X is a set, $r : X \times X \rightarrow X \times X$ is a solution and $a, b \in X$, then we denote

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

where λ_a, ρ_b are maps from X into itself.

We say that r is

- ▶ **left non-degenerate** if λ_a is bijective, for every $a \in X$;
- ▶ **right non-degenerate** if ρ_b is bijective, for every $b \in X$;
- ▶ **non-degenerate** if is both left and right non-degenerate.



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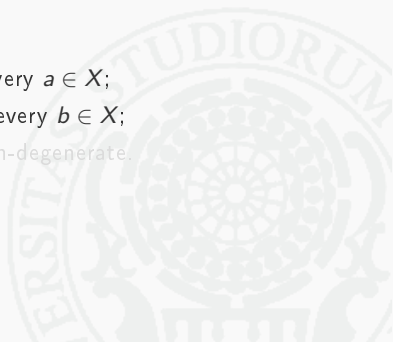
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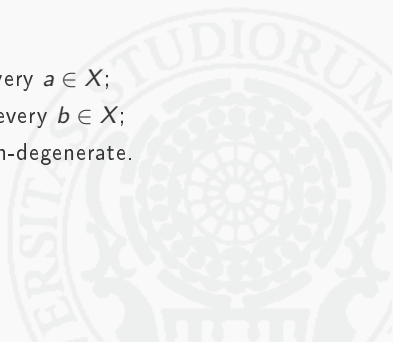
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An example - Gu Pei's solutions

If B is a group and $f : B \rightarrow B$ an endomorphism of B , then the map $r : B \times B \rightarrow B \times B$ defined by

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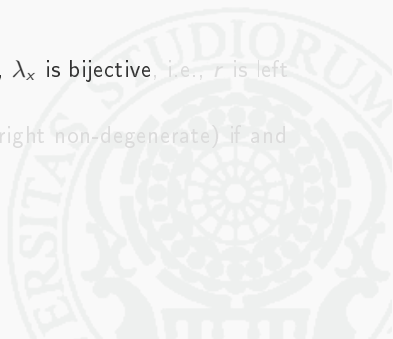
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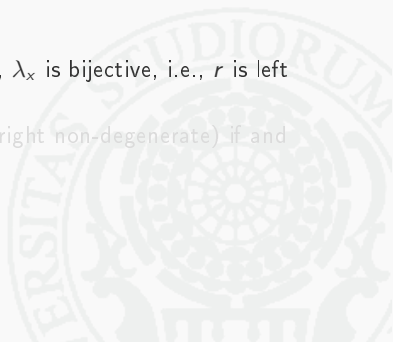
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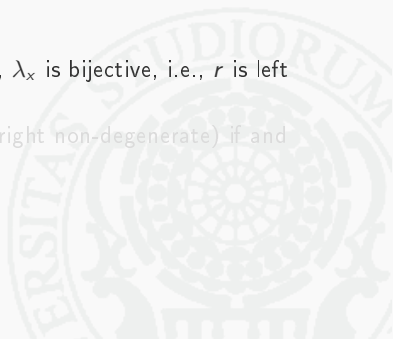
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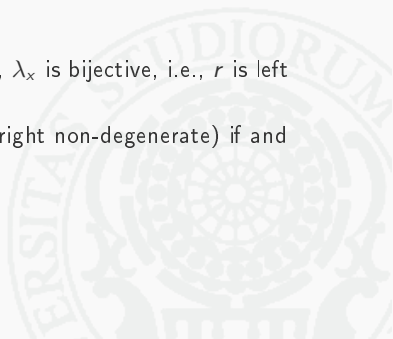
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Briefly, the state-of-the-art

In 1999 Etingof, Schedler and Soloviev, Gateva-Ivanova and Van den Bergh laid the groundwork for studying non-degenerate involutive solutions, mainly in group theory terms.

Many results are obtained for this class by several authors, such as Rump, Cedó, Jespers, Okniński and Smoktunowicz.

In 2000, Lu, Yan and Zhu and independently Soloviev started to study non-degenerate solutions not necessarily involutive. In 2017, Guarnieri and Vendramin obtained new results in this context.

We are going to focus on solutions that are only left non-degenerate. In particular, we will show how to determine such solutions through a new structure: the semi-brace.



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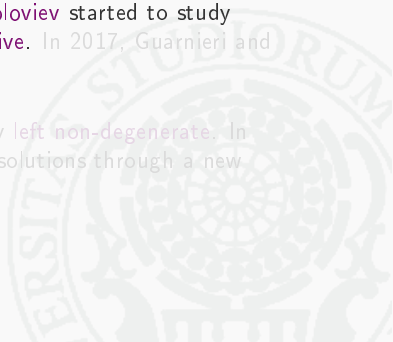
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Semi-braces

Definition (F. Catino, I. C., P. Stefanelli, J. Algebra, 2017)

If B is a set with two operations $+$ and \circ such that

- ▶ $(B, +)$ is a left cancellative semigroup,
- ▶ (B, \circ) is a group,
- ▶ $a \circ (b + c) = a \circ b + a \circ (a^- + c)$ holds for all $a, b, c \in B$, where a^- is the inverse of a with respect to the \circ ,

then $(B, +, \circ)$ is said a **(left) semi-brace**.

In particular, if B is a semi-brace with $(B, +)$ a group, then B is known as skew brace (Guarnieri and Vendramin, 2017) if in addition $(B, +)$ is abelian then B is a brace (Rump, 2007).

If (B, \circ) is a group and we set $a + b := b$, for all $a, b \in B$, then $(B, +, \circ)$ is a semi-brace that is not a skew brace.

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How to obtain a solution through a semi-brace

Theorem (F. Catino, I. C., P. Stefanelli, J. Algebra, 2017)

Let B be a semi-brace. Then, the map $r : B \times B \rightarrow B \times B$ given by

$$r(a, b) = \left(\underbrace{a \circ (a^- + b)}_{\lambda_a(b)}, \underbrace{(a^- + b)^- \circ b}_{r_b(a)} \right)$$

for all $a, b \in B$, is a left non-degenerate solution of the Yang-Baxter equation. We call r the solution associated to the semi-brace B .

Example. Let (B, \circ) be a group, f an endomorphism of (B, \circ) such that $f^2 = f$ and $(B, +, \circ)$ the semi-brace where $a + b = b \circ f(a)$, for all $a, b \in B$. Then, the associate solution to B is

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Hence the solution associated to a semi-brace is always left non-degenerate.

Moreover if $a, b, c \in B$, then

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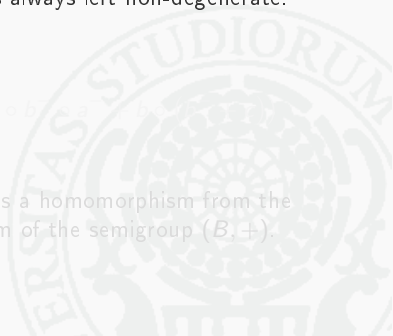
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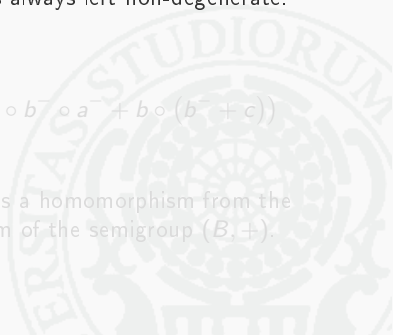
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If $a, b \in B$, we may prove that

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Sketch of the proof.

1. Note that

$$\lambda_a(b) \circ \rho_b(a) = a \circ (a^- + b) \circ (a^- + b)^- \circ b = a \circ b, \quad (*)$$

2. Compute $r_1 r_2 r_1(a, b, c) = (\lambda_{\lambda_a(b)} \lambda_{\rho_b(a)}(c), \rho_{\lambda_{\rho_b(a)}(c)} \lambda_a(b), \rho_c \rho_b(a))$ and $r_2 r_1 r_2(a, b, c) = (\lambda_a \lambda_b(c), \lambda_{\rho_{\lambda_b(c)}(a)} \rho_c(b), \rho_{\rho_c(b)} \rho_{\lambda_b(c)}(a))$.

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The solution associated to a skew brace

Theorem (Guarnieri, Vendramin, 2017)

Let $(B, +, \circ)$ be a skew brace. Then, the map $r : B \times B \rightarrow B \times B$ given by

$$r(a, b) = \left(\lambda_a(b), \lambda_{(\lambda_a(b))^{-1}}(-a \circ b + a + a \circ b) \right),$$

for all $a, b \in B$, is a non-degenerate bijective solution of the Yang-Baxter equation, where if $a \in B$, then

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Comparison between the two solutions

If B is a skew brace then B is a semi-brace with a group as additive structure.
 What is the relation between the solution associated to B as skew brace and the one associated to B as semi-brace?

Let $a, b \in B$. Then

$$\begin{aligned} -a + a \circ b &= -a + a \circ (0 + b) = -a + a \circ 0 + a \circ (a^{-1} + b) \\ &= -a + a + a \circ (a^{-1} + b) = a \circ (a^{-1} + b), \end{aligned}$$

i.e., the first components of the two solutions are the same.

Further

$$\begin{aligned} \lambda_{(\lambda_a(b))^{-1}}(-a \circ b + a + a \circ b) &= (a \circ (a^{-1} + b))^{-1} \circ (a \circ (a^{-1} + b) - a \circ b + a + a \circ b) \\ &= (a^{-1} + b)^{-1} \circ a^{-1} \circ (a \circ a^{-1} - a + a \circ b + a + a \circ b) \\ &= (a^{-1} + b)^{-1} \circ a^{-1} \circ (a \circ b + a + a \circ b) = (a^{-1} + b)^{-1} \circ a^{-1} \circ a \circ b = (a^{-1} + b)^{-1} \circ b = b. \end{aligned}$$

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Matched product (I)

Theorem (F. Catino, I.C., P. Stefanelli, in preparation)

Let G be a skew brace and E a trivial semi-brace, $\delta : G \rightarrow \text{Sym}(E)$ a right action of the group (G, \circ) on the set E and $\sigma : E \rightarrow \text{Aut}(G)$ a left action of the group (E, \circ) on the set G and σ_e is an automorphism of the group $(G, +)$, for every $e \in E$, such that

- ${}^e(g_1 \circ g_2) = ({}^e g_1) \circ ({}^{e\delta_1} g_2)$;
- $(e_1 \circ e_2)^\delta = e_1^{e_2 \delta} \circ e_2^\delta$;
- $0^\delta = 0$,

hold for all $g, g_1, g_2 \in G$ and $e, e_1, e_2 \in E$. Then the sum and the multiplication over the cartesian product $G \times E$ given by

$$(g_1, e_1) + (g_2, e_2) := (g_1 + g_2, e_2)$$

$$(g_1, e_1) \circ (g_2, e_2) := \left(g_1 \circ ((e_1^-)^{\delta_1})^-, e_1 \circ \left((e_2^-)^{(e_1^- g_1)^-} \right)^- \right)$$

define a structure of semi-brace, known as **matched product of G and E** (via δ and σ).

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Let G be a skew brace and E a trivial semi-brace, $\delta : G \rightarrow \text{Sym}(E)$ a right action of the group (G, \circ) on the set E and $\sigma : E \rightarrow \text{Aut}(G)$ a left action of the group (E, \circ) on the set G and σ_e is an automorphism of the group $(G, +)$, for every $e \in E$, such that

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Theorem (F. Catino, I.C., P. Stefanelli, in preparation)

Let B be a semi-brace, E the set of idempotents of $(B, +)$ and $G := B + 0$. Then G is a skew brace, E is a trivial semi-brace and there exist a right action of the group (G, \circ) on the set E and a left action of the group (E, \circ) on the set G and σ_e is an automorphism of the group $(G, +)$, for every $e \in E$, that satisfy the conditions in the previous theorem such that B is isomorphic to the matched product of G and E .

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Solution associated to the matched product

Let G be a skew brace, E a trivial semi-brace and B the matched product of G and E via actions δ and σ . If we denote by λ_g and ρ_g the maps λ_g and ρ_g in G and by $\bar{\lambda}_e$ the maps λ_e in E . Then the solution $r : B \times B \rightarrow B \times B$ is given by

$$\begin{aligned} r((g_1, e_1), (g_2, e_2)) \\ = \left(\left(\lambda_B \left(((e_1^{-1})^{e_1})^{-1} g_2 \right), \bar{\lambda}_{e_1} \left(e_2^{(\sigma_1 \circ e_1)^{-1} e_1} \right) \right), \left(\rho_B \left((e_1 \circ e_2)^{-1} e_1 \right) \right) \right) \end{aligned}$$

Hence the solution associated to B depends only on λ , ρ and $\bar{\lambda}$ actions.

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Thanks for your attention!

