Groups, Rings and the Yang-Baxter equation

UNIVERSITÀ DEL SALENTO

# Semi-braces and the Yang-Baxter equation

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Spa, June 22nd, 2017

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If $X$ is a set, a (set-theoretical) **solution** of the Yang-Baxter equation $r : X \times X \to X \times X$ is a map such that the well-known **braid equation**

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

is satisfied, where $r_1 = r \times \mathrm{id}_X$ and $r_2 = \mathrm{id}_X \times r$.

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How to obtain and construct all solutions of the Yang-Baxter equation?

Determining all set-theoretic solutions of the Yang-Baxter equation is a very difficult task. Even if we may find several works about this topic, it is still an open problem.

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# Solutions of the Yang-Baxter equation

In particular, if $X$ is a set, $r : X \times X \to X \times X$ is a solution and $a, b \in X$, then we denote

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

where $\lambda_a, \rho_b$ are maps from $X$ into itself.

We say that $r$ is

- **left non-degenerate** if $\lambda_a$ is bijective, for every $a \in X$;
- **right non-degenerate** if $\rho_b$ is bijective, for every $b \in X$;
- **non-degenerate** if is both left and right non-degenerate.

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# An example - Gu Pei's solutions

If $B$ is a group and $f : B \to B$ an endomorphism of $B$, then the map $r : B \times B \to B \times B$ defined by

$$r(x, y) = \left(xyf(x)^{-1}, f(x)\right)$$

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In particular, if $x, y \in B$

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# Briefly, the state-of-the-art

In 1999 Etingof, Schedler and Soloviev, Gateva-Ivanova and Van den Bergh laid the groundwork for studying non-degenerate involutive solutions, mainly in group theory terms.
Many results are obtained for this class by several authors, such as Rump, Cedó, Jespers, Okniński and Smoktunowicz.

In 2000, Lu, Yan and Zhu and independently Soloviev started to study non-degenerate solutions not necessarily involutive. In 2017, Guarnieri and Vendramin obtained new results in this context.

We are going to focus on solutions that are only left non-degenerate. In particular, we will show how to determine such solutions through a new structure: the **semi-brace**.

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# Semi-braces

**Definition (F. Catino, I. C., P. Stefanelli, J. Algebra, 2017)**

If $B$ is a set with two operations $+$ and $\circ$ such that

- $(B,+)$ is a left cancellative semigroup,
- $(B,\circ)$ is a group,
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then $(B,+,\circ)$ is said a **(left) semi-brace**.

In particular, if $B$ is a semi-brace with $(B,+)$ a group, then $B$ is known as skew brace (Guarnieri and Vendramin, 2017) if in addition $(B,+)$ is abelian then $B$ is a brace (Rump, 2007).

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# How to obtain a solution through a semi-brace

**Theorem (F. Catino, I. C., P. Stefanelli, J. Algebra, 2017)**

Let $B$ be a semi-brace. Then, the map $r : B \times B \to B \times B$ given by

$$r(a,b) = (\underbrace{a \circ (a^- + b)}_{\lambda_a(b)}, \underbrace{(a^- + b)^- \circ b}_{\rho_b(a)})$$

for all $a, b \in B$, is a left non-degenerate solution of the Yang-Baxter equation. We call $r$ the **solution associated to the semi-brace B**.

**Example.** Let $(B, \circ)$ be a group, $f$ an endomorphism of $(B, \circ)$ such that $f^2 = f$ and $(B, +, \circ)$ the semi-brace where $a + b = b \circ f(a)$, for all $a, b \in B$. Then, the associate solution to $B$ is

$$r(a,b) = \left(a \circ b \circ f(a)^-, f(a)\right),$$

for all $a, b \in B$. We may note that this solution belongs to the class of Gu Pei's solutions.

# How to obtain a solution through a semi-brace

**Theorem (F. Catino, I. C., P. Stefanelli, J. Algebra, 2017)**

Let $B$ be a semi-brace. Then, the map $r : B \times B \to B \times B$ given by

$$r\left(a,b\right) = \Big(\underbrace{a \circ \left(a^{-} + b\right)}_{\lambda_a(b)}, \underbrace{\left(a^{-} + b\right)^{-} \circ b}_{\rho_b(a)}\Big)$$

for all $a, b \in B$, is a left non-degenerate solution of the Yang-Baxter equation. We call $r$ the **solution associated to the semi-brace B.**

**Example.** Let $(B, \circ)$ be a group, $f$ an endomorphism of $(B, \circ)$ such that $f^2 = f$ and $(B, +, \circ)$ the semi-brace where $a + b = b \circ f\left(a\right)$, for all $a, b \in B$. Then, the associate solution to $B$ is

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Hence, if $B$ is a semi-brace that is not a skew brace, then $|E| \geq 2$ and $\rho_b$ is not bijective, for every $b \in B$, i.e., the solution associated to $B$ is right degenerate.

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# Sketch of the proof.

1. Note that

$$\lambda_a(b) \circ \rho_b(a) = a \circ (a^- + b) \circ (a^- + b)^- \circ b = a \circ b, \qquad (*)$$

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# The solution associated to a skew brace

**Theorem (Guarnieri, Vendramin, 2017)**

Let $(B, +, \circ)$ be a skew brace. Then, the map $r : B \times B \to B \times B$ given by

$$r\left(a,\ b\right) = \left(\lambda_a\left(b\right),\ \lambda_{\left(\lambda_a\left(b\right)\right)^-}\left(-a \circ b + a + a \circ b\right)\right),$$

for all $a, b \in B$, is a non-degenerate bijective solution of the Yang-Baxter equation, where if $a \in B$, then

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# Comparison between the two solutions

If $B$ is a skew brace then $B$ is a semi-brace with a group as additive structure.
What is the relation between the solution associated to $B$ as skew brace and the one associated to $B$ as semi-brace?

Let $a, b \in B$. Then

$$\begin{aligned}-a + a \circ b = -a + a \circ (0 + b) &= -a + a \circ 0 + a \circ \left(a^{-} + b\right) \\ &= -a + a + a \circ \left(a^{-} + b\right) = a \circ \left(a^{-} + b\right),\end{aligned}$$

i.e., the first components of the two solutions are the same.
Further

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$$\begin{aligned} &\lambda_{(\lambda_a(b))^-}(-a \circ b + a + a \circ b) \\ &= \left(a \circ \left(a^- + b\right)\right)^- \circ \left(a \circ \left(a^- + b\right) - a \circ b + a + a \circ b\right) \\ &= \left(a^- + b\right)^- \circ a^- \circ \left(a \circ a^- - a + a \circ b - a \circ b + a + a \circ b\right) \\ &= \left(a^- + b\right)^- \circ a^- \circ \circ a \circ b = \left(a^- + b\right)^- \circ b = \rho_b(a), \end{aligned}$$

i.e., also the second components are the same.

# Comparison between the two solutions

If $B$ is a skew brace then $B$ is a semi-brace with a group as additive structure. What is the relation between the solution associated to $B$ as skew brace and the one associated to $B$ as semi-brace?

Let $a, b \in B$. Then

$$\begin{aligned} -a + a \circ b = -a + a \circ (0 + b) &= -a + a \circ 0 + a \circ \left(a^- + b\right) \\ &= -a + a + a \circ \left(a^- + b\right) = a \circ \left(a^- + b\right), \end{aligned}$$

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Let $G$ be a skew brace and $E$ a trivial semi-brace, $\delta : G \to \mathrm{Sym}(E)$ a right action of the group $(G, \circ)$ on the set $E$ and $\sigma : E \to \mathrm{Aut}(G)$ a left action of the group $(E, \circ)$ on the set $G$ and $\sigma_e$ is an automorphism of the group $(G, +)$, for every $e \in E$, such that

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**Theorem (F. Catino, I.C., P. Stefanelli, in preparation)**

Let $B$ be a semi-brace, $E$ the set of idempotents of $(B, +)$ and $G := B + 0$. Then $G$ is a skew brace, $E$ is a trivial semi-brace and there exist a right action of the group $(G, \circ)$ on the set $E$ and a left action of the group $(E, \circ)$ on the set $G$ and $\sigma_e$ is an automorphism of the group $(G, +)$, for every $e \in E$, that satisfy the conditions in the previous theorem such that $B$ is isomorphic to the matched product of $G$ and $E$.

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# Solution associated to the matched product

Let $G$ be a skew brace, $E$ a trivial semi-brace and $B$ the matched product of $G$ and $E$ via actions $\delta$ and $\sigma$. If we denote by $\lambda_g$ and $\rho_g$ the maps $\lambda_g$ and $\rho_g$ in $G$ and by $\bar{\lambda}_e$ the maps $\lambda_e$ in $E$. Then the solution $r: B \times B \to B \times B$ is given by

$$r\left(\left(g_1, e_1\right),\ \left(g_2, e_2\right)\right)$$
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Hence the solution associated to $B$ depends only on $\lambda_g$, $\rho_g$, $\bar{\lambda}_e$ and on the actions.

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$$r\left((g_1, e_1),\ (g_2, e_2)\right) = \left(\left(\lambda_{g_1}\left(\left(\left(e_1^-\right)^{g_1}\right)^- g_2\right),\ \bar{\lambda}_{e_1}\left(e_2^{(e_1 \circ e_2)^- g_1}\right)\right),\ \left(\rho_{g_2}\left({}^{(e_1 \circ e_2)^-} g_1^-\right),\ 0_E\right)\right)$$

Hence the solution associated to $B$ depends only on $\lambda_g$, $\rho_g$, $\bar{\lambda}_e$ and on the actions.

Further, note that $0_E = \rho_{e_2}(e_1)$, for all $e_1, e_2 \in E$.

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Thanks for your attention!