Regular subgroups of an affine group

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GROUP THEORY IN FLORENCE: A MEETING IN HONOUR OF GUIDO ZAPPA

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Let $V$ be a vector space over a field $F$. The **affine group** $AGL(V)$ of $V$ is the group generated by $GL(V)$ and $T(V)$, $AGL(V) := GL(V) \rtimes T(V)$, where $GL(V)$ is the group of invertible linear maps of $V$ and $T(V)$ is the translation group of $V$.

A permutation group $G$ over a set $X$ is called **regular** if, for all $x, y \in X$, there exists a unique $\pi \in G$ such that $x\pi = y$.

Clearly $T(V)$ and its conjugated subgroups by an element of $GL(V)$ are abelian regular subgroups of $AGL(V)$.

**Problem [M. W. Liebeck, C. E. Praeger, J. Saxl, 2009]**

Find all regular subgroups of $AGL(V)$.
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The affine group of a vector space

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In 2006, A. Caranti, F. Dalla Volta and M. Sala obtained a simple description of all abelian regular subgroups of the affine group $AGL(V)$ in terms of radical commutative associative $F$-algebras that have $V$ as underlying vector space.

In 2009, F. Catino and R. Rizzo generalized this result obtaining a complete description of all regular subgroups of the affine group $AGL(V)$ in terms of radical $F$-braces that have $V$ as underlying vector space.
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In 2009, F. Catino and R. Rizzo generalized this result obtaining a complete description of all regular subgroups of the affine group $AGL(V)$ in terms of radical $F$-braces that have $V$ as underlying vector space.
$F$-brace definition

Definition ($F$-brace)

Let $V$ be a vector space over a field $F$ and let $\cdot$ an operation on $V$. We call $V^\bullet := (V, +, \cdot)$ an $F$-brace if, for all $x, y, z \in V$ and for all $\lambda \in F$, the following conditions hold:

1. $(x + y) \cdot z = x \cdot z + y \cdot z$;
2. $x \cdot (y + z + y \cdot z) = x \cdot y + x \cdot z + (x \cdot y) \cdot z$
3. $\lambda(x \cdot y) = (\lambda x) \cdot y$.

Clearly every associative $F$-algebra is a $F$-brace.

On the other hand, every commutative $F$-brace is an associative $F$-algebra.
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Definition (F-brace)

Let $V$ be a vector space over a field $F$ and let $\cdot$ an operation on $V$. We call $V^* := (V, +, \cdot)$ an F-brace if, for all $x, y, z \in V$ and for all $\lambda \in F$, the following conditions hold:

1. $(x + y) \cdot z = x \cdot z + y \cdot z$;
2. $x \cdot (y + z + y \cdot z) = x \cdot y + x \cdot z + (x \cdot y) \cdot z$
3. $\lambda(x \cdot y) = (\lambda x) \cdot y$.

Clearly every associative $F$-algebra is a $F$-brace.

On the other hand, every commutative $F$-brace is an associative $F$-algebra.
Let $V$ be an $F$-brace, as for associative $F$-algebras, we may introduce the \textit{adjoint operation} on $V$ setting

$$u \circ v := u + v + u \cdot v,$$

for all $u, v \in V$.

In general $(V, \circ)$ is a semigroup. If $(V, \circ)$ is a group, we say that the $F$-brace $V$ is \textit{radical}.
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In general $(V, \circ)$ is a semigroup. If $(V, \circ)$ is a group, we say that the $F$-brace $V$ is \textit{radical}.
If $V$ is a radical $F$-brace and $\circ$ is the adjoint operation then $(V, +)$ is a vector space over $F$, $(V, \circ)$ is a group and the conditions

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(v + w) \circ z + z = v \circ z + w \circ z \quad (1)
$$

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\lambda (v \circ w) = (\lambda v) \circ w + (\lambda - 1) w. \quad (2)
$$

hold for all $v, w, z \in V$ and for all $\lambda \in F$. Conversely, if $(V, +)$ and $(V, \circ)$ are a vector space over $F$ and a group respectively that satisfy equations (1) and (2), then posed $v \cdot w := v \circ w - v - w$ for all $v, w \in V$, we have that $(V, +, \cdot)$ is an radical $F$-brace.
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Theorem (Catino, Rizzo, 2009)

Let $V$ be a vector space over a field $F$. Denote by $\mathcal{RB}$ the class of radical $F$-brace with underlying vector space $V$ and by $T$ the set of all regular subgroups of the affine group $AGL(V)$.

1. Let $V^* \in \mathcal{RB}$. Then
   \[
   T(V^*) = \{ \tau_x | x \in V \},
   \]
   where $\tau_x : V \to V$, $y \mapsto y \circ x$, is a regular subgroup of the affine group $AGL(V)$.

2. The map
   \[
   f : \mathcal{RB} \to T, \quad V^* \mapsto T(V^*)
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   is a bijection.

In this correspondence, isomorphism classes of $F$-brace correspond to conjugacy classes under the action of $GL(V)$ of regular subgroups of $AGL(V)$. 
Theorem (Catino, Rizzo, 2009)

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Link between radical $F$-braces and regular subgroups (I)

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In this correspondence, isomorphism classes of $F$-brace correspond to conjugacy classes under the action of $GL(V)$ of regular subgroups of $AGL(V)$.
Let $F$ be a field, $n \in \mathbb{N}$ and $V := F^n$. By immersion of $AGL(n, F)$ in $GL(n + 1, F)$ and by previous Theorem we have that if

$$T = \left\{ \begin{pmatrix} 1 & v \\ 0 & \gamma_v \end{pmatrix} \bigg| v \in F^n \right\}$$

is a regular subgroup of $AGL(n, F)$, then there exists a unique radical $F$-brace $V^\bullet$ such that $T = T(V^\bullet)$.

Moreover the adjoint operation on $V^\bullet$ is

$$v \circ w = v \gamma_w + w = v \gamma_w t_w = v \tau_w,$$

for all $v, w \in V$.

That is, fixed a regular subgroup $T$ of the affine group $AGL(n, F)$, there exists a unique radical $F$-brace on $F^n$ such that its adjoint group is isomorphic to $T$. 
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That is, fixed a regular subgroup $T$ of the affine group $AGL(n, F)$, there exists a unique radical $F$-brace on $F^n$ such that its adjoint group is isomorphic to $T$. 
For example, let $F$ be a field and let $\tau$ an endomorphism of the additive group of $F$. It is easy to see that the group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x\tau \\ 0 & 0 & 1 \end{pmatrix} \bigg| \ x, y \in F \right\}$$

(3)

is a regular subgroup of the affine group $AGL(2, F)$

By previous result this, is in correspondence with the $F$-brace with underling vector space $F^2$ such that

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) \circ (x_2, y_2) = (x_1, y_1) \begin{pmatrix} 1 & x_2\tau \\ 0 & 1 \end{pmatrix} + (x_2, y_2) = (x_1 + x_2, x_1(x_2\tau) + y_1 + y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in F^2$. 
Conversely, if $V^\bullet$ is a radical $F$-brace, set $\gamma_w : V \to V$, $v \mapsto v \circ w - w$, for all $w \in V$, and consider

$$T = \left\{ \begin{pmatrix} 1 & v \\ 0 & \gamma_v \end{pmatrix} \middle| v \in V \right\},$$

it is a regular subgroup of $AGL(n, F)$ and it is isomorphic to the adjoint group of the radical $F$-brace $V^\bullet$. 
Conversely, if $V^\bullet$ is a radical $F$-brace, set $\gamma_w : V \to V$, $v \mapsto v \circ w - w$, for all $w \in V$, and consider

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it is a regular subgroup of $AGL(n, F)$ and it is isomorphic to the adjoint group of the radical $F$-brace $V^\bullet$. 
Link between radical $F$-brace and regular subgroup (IV)

For example, if we consider the previous $F$-brace with underlying vector space $F^2$ such that

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$$(x_1, y_1) \circ (x_2, y_2) = (x_1 + x_2, x_1(x_2 \tau) + y_1 + y_2)$$

with $\tau$ an endomorphism of the additive group of $F$ and $x_1, x_2, y_1, y_2 \in F$ we may compute the regular subgroup of $AGL(2, F)$ corresponding to this $F$-brace.

In fact, let $\{(1, 0), (0, 1)\}$ be the canonical basis of $F^2$. Then, for all $x, y$

$$(1, 0) \gamma_{(x,y)} = (1, 0) \circ (x, y) - (x, y) = (1 + x, x \tau + y) - (x, y) = (1, x \tau)$$

$$(0, 1) \gamma_{(x,y)} = (0, 1) \circ (x, y) - (x, y) = (x, 1 + y) - (x, y) = (0, 1)$$

and so the regular subgroup is

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \tau \\ 0 & 0 & 1 \end{pmatrix} \bigg| x, y \in F \right\},$$

the initial regular subgroup.
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A cohomological approach

We introduce some cohomological tools in analogy with the method employed by W. A. de Graaf for the classification of nilpotent associative algebras of dimensions 2 and 3 over any field. In particular, we translate the concepts of “2-cocycles” and the “Hochschild product” from the context of associative $F$-algebras into that of $F$-braces.

In this way we obtain a description of all finite dimensional radical $F$-braces with non-trivial annihilator.
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In this way we obtain a description of all finite dimensional radical $F$-braces with non-trivial annihilator.
We define the set of **right annihilator** of an $F$-brace $V$ and that of **left annihilator** respectively as follows:

\[
Ann_R(V) := \{ x \mid x \in V, \forall v \in V, \forall \lambda \in F \ v \cdot (\lambda x) = 0 \}
\]

and

\[
Ann_L(V) := \{ x \mid x \in V, \forall v \in V, x \cdot v = 0 \}.
\]

These sets are subspaces of $V$. Note that the previous definitions of left and right annihilator cannot be symmetric, since in general, if $v, w \in V$ and $\lambda \in F$, then $v \cdot (\lambda w) \neq \lambda(v \cdot w)$.

The set $Ann(V) := Ann_L(V) \cap Ann_R(V)$ is called the **annihilator** of the $F$-brace $V$. 
The annihilator of an $F$-brace

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2-cocycles of $F$-braces

**Definition**

Let $A$ be an $F$-brace and $V$ a vector space over a field $F$. A map $\theta : A \times A \to V$ with the properties:

1. $(\lambda a + \mu b, c)\theta = \lambda ((a, c)\theta) + \mu ((b, c)\theta)$,
2. $(a, b + c + b \cdot c)\theta = (a, b)\theta + (a, c)\theta + (a \cdot b, c)\theta$,

for all $a, b, c \in A$ and $\lambda, \mu \in F$, is called a 2-cocycle of $A$ with values in $V$.

Thus 2-cocycles of $F$-algebras [see for instance, R. S. Pierce, *Associative Algebras*] are particular cases of 2-cocycles of $F$-braces. But if we regard an $F$-algebra as an $F$-brace, then a 2-cocycle in the sense of the previous definition is not necessarily a 2-cocycle in the usual sense.
2-cocycles of $F$-braces

**Definition**

Let $A$ be an $F$-brace and $V$ a vector space over a field $F$. A map $\theta : A \times A \rightarrow V$ with the properties:

1. $(\lambda a + \mu b, c)\theta = \lambda((a, c)\theta) + \mu((b, c)\theta)$,
2. $(a, b + c + b \cdot c)\theta = (a, b)\theta + (a, c)\theta + (a \cdot b, c)\theta$,

for all $a, b, c \in A$ and $\lambda, \mu \in F$, is called a 2-cocycle of $A$ with values in $V$.

Thus 2-cocycles of $F$-algebras [see for instance, R. S. Pierce, *Associative Algebras*] are particular cases of 2-cocycles of $F$-braces. But if we regard an $F$-algebra as an $F$-brace, then a 2-cocycle in the sense of the previous definition is not necessarily a 2-cocycle in the usual sense.
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Thus 2-cocycles of $F$-algebras [see for instance, R. S. Pierce, *Associative Algebras*] are particular cases of 2-cocycles of $F$-braces. But if we regard an $F$-algebra as an $F$-brace, then a 2-cocycle in the sense of the previous definition is not necessarily a 2-cocycle in the usual sense.
Let $N$ be the zero $F$-algebra of dimension $n$ over a field $F$ and $\tau$ an endomorphism of the additive group of $F$. Then the map

$$\theta : N \times N \rightarrow F, \left( \sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i \right) \mapsto \left( \sum_{i=1}^{n} x_i \right) \left( \sum_{i=1}^{n} y_i \right) \tau$$

is a 2-cocycle of the $F$-brace $N$ but, in general, not of the $F$-algebra $N$.

In particular, $\theta$ is a 2-cocycle of the $n$-dimensional zero $F$-algebra if and only if $\tau$ is linear.
Example

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Hochschild product

Definition

Let \( A \) be an \( F \)-brace, \( V \) an \( F \)-vector space, \( \theta : A \times A \to V \) a 2-cocycle. Put \( A_\theta := A \oplus V \). For all \( a, b \in A \) and \( v, w \in V \) we define

\[
(a + v) \cdot (b + w) := a \cdot b + (a, b)\theta.
\]

(5)

The \( F \)-brace \( A_\theta \) is called a **Hochschild product** of \( A \) by \( V \).

What may we say of \( A_\theta \) if \( A \) is a radical \( F \)-brace?

If \( A \) is a radical \( F \)-brace and \( \theta \) is a 2-cocycle of \( A \) with values in an \( F \)-vector space \( V \). Then \( A_\theta \) is radical.
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Let $B$ be a radical $F$-brace such that $\text{Ann}(B) \neq \{0\}$. Then there exist an $F$-brace $A$, an $F$-vector space $V$ and a 2-cocycle $\theta : A \times A \to V$ such that $B$ is isomorphic to $A_\theta$.

We consider $A := B/\text{Ann}(B)$, $V := \text{Ann}(B)$.

If $\pi : B \to A$ be the projection map and we choose a linear map $\sigma : A \to B$ such that $(x\sigma)\pi = x$, for all $x \in A$, then we obtain a function $\theta$ from $A \times A$ into $V$ by defining

$$ (x, y)\theta := x\sigma \cdot y\sigma - (x \cdot y)\sigma. \quad (6) $$

that is a 2-cocycle.
Description of F-braces with non-trivial annihilator (I)


Let $B$ be a radical $F$-brace such that $\text{Ann}(B) \neq \{0\}$. Then there exist an $F$-brace $A$, an $F$-vector space $V$ and a 2-cocycle $\theta : A \times A \rightarrow V$ such that $B$ is isomorphic to $A_\theta$.

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The regular subgroups of an affine group obtained by the Hochschild product of radical $F$-braces have non-trivial intersection with the translation group.

**Proposition (F. Catino, R. Rizzo, 2009)**

Let $V^*$ be a radical $F$-brace with underlying vector space $V$ over a field $F$. Let $T(V^*) = \{(\gamma_a, a) \mid a \in V\}$ and $T(V)$ be the translation group. Then

$$T(V) \cap T(V^*) = \{(\gamma_a, a) \mid a \in \text{Soc}(V^*)\},$$

where $\text{Soc}(V^*) := \{x \mid x \in V, \forall v \in V \ v \cdot x = 0\}$ is the socle of $V$.

Let us note that $\text{Ann}_R(V) \subseteq \text{Soc}(V)$.

In particular, if $V^*$ is a radical associative $F$-algebra of finite dimension, then $T(V) \cap T(V^*) \neq 1$. 
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### Proposition (F. Catino, R. Rizzo, 2009)

Let $V^\bullet$ be a radical $F$-brace with underlying vector space $V$ over a field $F$. Let $T(V^\bullet) = \{ (\gamma_a, a) \mid a \in V \}$ and $T(V)$ be the translation group. Then

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The intersection with the translation group (I)

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The intersection with the translation group (II)

The Hochschild Product of radical $F$-braces is not exhaustive because there are examples of subgroups of affine group with trivial intersection with the translations as the following:

**Theorem (P. Hegedűs, 2000)**

*Let $p$ be a prime. If $p = 2$ then assume $n = 3$, or $n \geq 5$. If $p$ is odd then assume $n \geq 4$. Then the affine group $AGL(n, p)$ has a regular subgroup which contains no translations other than the identity.*
Let $p$ be a prime and $n \in \mathbb{N}$, if $p$ is odd, let $n \geq 4$ otherwise if $p = 2$, let $n \geq 3$ odd. Over the field $\mathbb{F}_p$ of $p$ elements. Consider the immersion of $AGL(n, p)$ into $GL(n + 1, p)$ and let

- $q : \mathbb{F}_p^{n-1} \to \mathbb{F}_p$ be a non-degenerate quadratic form;
- $b : \mathbb{F}_p^{n-1} \times \mathbb{F}_p^{n-1} \to \mathbb{F}_p$ be the symmetric bilinear form associated to $q$;
- $X$ the matrix associated to $b$ respect to a fixed basis $((v + w)q = vq + wq + vXw^T$ for all $v, w \in \mathbb{F}_p^{n-1}$);
- $A$ the orthogonal non-singular matrix $(n - 1) \times (n - 1)$ of order $p$ ($XA^T = A^{-1}X$) and such that $vq = (vA)q$ for all $v \in \mathbb{F}_p^{n-1}$.
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Hegedűs’ subgroups (I)

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Ilaria Colazzo: Regular subgroups of an affine group
Hegedűs’ subgroups (II)

Then the set

\[ T := \left\{ \left( \begin{array}{ccc} 1 & m & v \\ 0 & 1 & 0 \\ 0 & A^m X w^T & A^m \end{array} \right) \middle| m \in \mathbb{F}_p, v \in \mathbb{F}_{p}^{n-1} \right\}. \]

is a regular subgroup of the affine group $AGL(n, p)$ that has trivial intersection with the translation group.

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Let

- $F$ be a field;
- $V$ be a $n$-dimensional vector space over $F$ (reviewed as zero $F$-brace);
- $q : V \rightarrow F$ be a quadratic form and $b : V \times V \rightarrow F$ the polar form of $q$;
- $\alpha : F \rightarrow \text{Aut}(V)$ be a group homomorphism from $(F, +)$ to the automorphism group of the $F$-brace $V$.

If

$$(v^s)q = (v)q$$

holds for all $v \in V$ and $s \in F$, then we may define over $F \times V$ a structure of radical $F$-brace.
The Asymmetric Product of zero $F$-braces (I)

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The Asymmetric Product of zero $F$-braces (II)

Set the sum, the multiplication over $F \times V$ and the scalar multiplication

\begin{align*}
(s_1, v_1) + (s_2, v_2) &= (s_1 + s_2 + (v_1, v_2)b, v_1 + v_2) \\
(s_1, v_1) \circ (s_2, v_2) &= (s_1 \circ s_2, v_1^{s_2} + v_2) \\
\lambda(s_1, v_1) &= (\lambda s_1 + \lambda(\lambda - 1)(v_1)q, \lambda v_1)
\end{align*}

for all $(s_1, v_1), (s_2, v_2) \in S \times V$ and $\lambda \in F$. This radical $F$-brace is called the \textit{Asymmetric Product} of $V$ by $F$ (denoted by $F \triangleleft \circ V$).

In this cases we may check the intersection of the regular subgroup associated to the radical $F$-brace with the translation group through properties of $b$ and $\alpha$.

The regular subgroup of $AGL(n + 1, F)$ associated with $F \triangleleft \circ V$ intersects trivially the translation group if and only if the symmetric bilinear form $b : V \times V \to F$ (the quadratic form $q : V \to F$, respectively) is non-degenerate and the action $\alpha : F \to \text{Aut}(V)$ is faithful.
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\end{align}

for all $(s_1, v_1), (s_2, v_2) \in S \times V$ and $\lambda \in F$. This radical $F$-brace is called the Asymmetric Product of $V$ by $F$ (denoted by $F \times_v V$).

In this cases we may check the intersection of the regular subgroup associated to the radical $F$-brace with the translation group through properties of $b$ and $\alpha$.

The regular subgroup of $AGL(n + 1, F)$ associated with $F \times_v V$ intersects trivially the translation group if and only if the symmetric bilinear form $b : V \times V \to F$ (the quadratic form $q : V \to F$, respectively) is non-degenerate and the action $\alpha : F \to \text{Aut}(V)$ is faithful.
The Asymmetric Product of zero $F$-braces (II)

Set the sum, the multiplication over $F \times V$ and the scalar multiplication

\[(s_1, v_1) + (s_2, v_2) = (s_1 + s_2 + (v_1, v_2)b, v_1 + v_2) \quad (9)\]
\[(s_1, v_1) \circ (s_2, v_2) = (s_1 \circ s_2, v_1^{s_2} + v_2) \quad (10)\]
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Let $p$ be a prime and $n \geq 4$ (if $p$ is odd) or $n \geq 4$ even (if $p = 2$). Set $V = \mathbb{F}_{p^m}$ and consider

1. $q : V \to \mathbb{F}_{p^m}$ an isotropic quadratic form such that its polar form $b$ is non-degenerate.
2. $A_1, \ldots, A_m \in O(V, q) = \{ A \in GL(n, p^m) \mid \forall v \in \mathbb{F}_{p^m} \quad vq = (vA)q \}$ distinct of order $p$ that commutes two by two.
3. $\alpha : \mathbb{F}_{p^m} = \bigoplus_{i=1}^m \langle \omega_i \rangle \longrightarrow GL(n, \mathbb{F}_{p^m})$ a group homomorphism such that $\omega_i \alpha = A_i$ ($\alpha$ is clearly injective)

Then $b$ and $\alpha$ are compatible and we may consider the radical brace over $\mathbb{F}_{p^m}$ asymmetric product $\mathbb{F}_{p^m} \bowtie V$. 
Let $p$ be a prime and $n \geq 4$ (if $p$ is odd) or $n \geq 4$ even (if $p = 2$). Set $V = \mathbb{F}^n_{p^m}$ and consider

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Generalization of Hegedűs’ subgroups (I)

Let $p$ be a prime and $n \geq 4$ (if $p$ is odd) or $n \geq 4$ even (if $p = 2$). Set $V = \mathbb{F}_{p^m}^n$ and consider

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Then \( b \) and \( \alpha \) are compatible and we may consider the radical brace over \( \mathbb{F}_{p^m} \) asymmetric product \( \mathbb{F}_{p^m} \ltimes \circ V \).
Let $X$ the matrix associated to $-b$ respect a fixed basis, then the regular subgroup of the affine group associated to this $\mathbb{F}_{p^m}$-brace is given by

$$G = \left\{ \begin{pmatrix} 1 & \sum_{i=1}^{m} \mu_i \omega_i + (v_2) \cdot q & v_2 \\ 0 & 1 & 0 \\ 0 & \left( \prod_{i=1}^{m} A_i^{\mu_i} \right) X v_2^T & \prod_{i=1}^{m} A_i^{\mu_i} \end{pmatrix} \left| \sum_{i=1}^{m} \mu_i \omega_i, v_2 \right) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}^n \right\}.$$ 

If $m = 1$ then $G$ becomes

$$G = \left\{ \begin{pmatrix} 1 & \mu + (v_2) \cdot q & v_2 \\ 0 & 1 & 0 \\ 0 & A^\mu X v_2^T & A^\mu \end{pmatrix} \left| (m, v_2) \in \mathbb{F}_p \times \mathbb{F}_p^n \right\}.$$ 

that is isomorphic to the regular subgroup constructed by Hegedűs.
Generalization of Hegedűs’ subgroups (II)

Let $X$ the matrix associated to $-b$ respect a fixed basis, then the regular subgroup of the affine group associated to this $\mathbb{F}_{p^m}$-brace is given by

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If $m = 1$ then $G$ becomes

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Thank you for your attention.