The solutions of the Yang-Baxter equation with finite order

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Introduction

Studying the solutions of the Yang-Baxter equation has been an important research area for the past 50 years. Indeed the Yang-Baxter equation is a fundamental tool in several different fields of research such as statistical mechanics, quantum group theory, and low-dimensional topology. In 1992, Drinfel′d initiated the study of a specific class of solutions, named set-theoretical solutions. Given a set X, a set-theoretical solution of the Yang-Baxter equation (or shortly a solution) is a map $r : X \times X \to X \times X$ such that the following condition is satisfied

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

where $r_1 = r \times \mathrm{id}_X$ and $r_2 = \mathrm{id}_X \times r$.
If r is such a solution on X, for $x, y \in X$, let's define the maps $\lambda_x : X \to X$ and $\rho_y : X \to X$ by $r(x, y) = (\lambda_x(y), \rho_y(x))$. A solution is said to be of finite order if there exist two non-negative integers p, i such that $r^{p+i} = r^i$. It is worth pointing out that the most extensively studied solutions, namely

- the involutive solutions ($r^2 = \mathrm{id}$)
- the finite bijective solutions ($r^n = \mathrm{id}$)
- the idempotent solutions ($r^2 = r$)

are all of finite order.

Aim

To show how the matched product of solutions is a unifying tool for treating solutions of finite order.

The matched product of solutions

We introduce a new construction technique for solutions of the Yang-Baxter equation that allows one to obtain new solutions on the cartesian product of sets, starting from completely arbitrary solutions.
Given a solution r_S on a set S and a solution r_T on a set T, if $\alpha : T \to \mathrm{Sym}(S)$ and $\beta : S \to \mathrm{Sym}(T)$ are maps, set $\alpha_u := \alpha(u)$, for every $u \in T$, and $\beta_a := \beta(a)$, for every $a \in S$, then the quadruple $(r_S, r_T, \alpha, \beta)$ is said to be a *matched product system of solutions* if the following conditions hold

$$\alpha_u\alpha_v = \alpha_{\lambda_u(v)}\alpha_{\rho_v(u)} \quad \text{(s1)} \qquad \beta_a\beta_b = \beta_{\lambda_a(b)}\beta_{\rho_b(a)} \quad \text{(s2)}$$

$$\rho_{\alpha_u^{-1}(b)}\alpha^{-1}_{\beta_a(u)}(a) = \alpha^{-1}_{\beta_{\rho_b(a)}\beta_b^{-1}(u)}\rho_b(a) \quad \text{(s3)} \qquad \rho_{\beta_a^{-1}(v)}\beta^{-1}_{\alpha_u(a)}(u) = \beta^{-1}_{\alpha_{\rho_v(u)}\alpha_v^{-1}(a)}\rho_v(u) \quad \text{(s4)}$$

$$\lambda_a\alpha_{\beta_a^{-1}(u)} = \alpha_u\lambda_{\alpha_u^{-1}(a)} \quad \text{(s5)} \qquad \lambda_u\beta_{\alpha_u^{-1}(a)} = \beta_a\lambda_{\beta_a^{-1}(u)} \quad \text{(s6)}$$

for all $a, b \in S$ and $u, v \in T$. Any matched product system of solutions determines a new solution on the set $S \times T$: If $(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions, then the map $r : S{\times}T \times S{\times}T \to S{\times}T \times S{\times}T$ defined by

$$r((a,u),(b,v)) := ((\underbrace{\alpha_u\lambda_{\bar{a}}(b)}_{\substack{\shortparallel \\ A}},\ \underbrace{\beta_a\lambda_{\bar{u}}(v)}_{\substack{\shortparallel \\ U}}),\ (\alpha_{\bar{U}}^{-1}\rho_{\alpha_{\bar{u}}(b)}(a),\ \beta_{\bar{A}}^{-1}\rho_{\beta_{\bar{a}}(v)}(u)))$$

for all $(a,u), (b,v) \in S \times T$, is a solution. This solution is called the matched product of the solutions r_S and r_T (via α and β) and it is denoted by $r_S \bowtie r_T$. Here, if $(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions, we denote $\alpha_u^{-1}(a)$ with $\bar{a}$ and $\beta_a^{-1}(u)$ with $\bar{u}$, when the pair $(a,u) \in S \times T$ is clear from the context.

The matched product of the solutions of finite order

The key result is that the matched product preserves the property to of solutions to be of finite order.

Theorem

Let $(r_S, r_T, \alpha, \beta)$ be a matched product system of solutions. Then, the solutions r_S and r_T are of finite order if and only if the solution $r_S \bowtie r_T$ is of finite order.

Furthermore, determining the order of the matched product of two solutions of finite order requires the notion of index and period. We recall that the index and the period of any solution r of finite order are defined as

$$\mathrm{i}(r) := \min\left\{j \mid\ j \in \mathbb{N}_0, \exists l \in \mathbb{N}\ r^l = r^j\right\},$$

$$\mathrm{p}(r) := \min\left\{k \mid\ k \in \mathbb{N}, r^{k+\mathrm{i}(r)} = r^{\mathrm{i}(r)}\right\}.$$

Theorem

Let $(r_S, r_T, \alpha, \beta)$ be a matched product system of solutions. If r_S and r_T are solutions of finite order, then

$$\mathrm{i}(r_S \bowtie r_T) = \max\{\mathrm{i}(r_S), \mathrm{i}(r_T)\}$$

$$\mathrm{p}(r_S \bowtie r_T) = \mathrm{lcm}(\mathrm{p}(r_S), \mathrm{p}(r_T)).$$

The index and the period of the matched product solution $r_S \bowtie r_T$ give us upper bounds of the indexes and periods of r_S and r_T. Indeed if $(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions and $r_S \bowtie r_T$ is a solution of finite order then

$$\mathrm{p}(r_S) \mid \mathrm{p}(r_S \bowtie r_T) \quad \text{and} \quad \mathrm{p}(r_T) \mid \mathrm{p}(r_S \bowtie r_T)$$
$$\mathrm{i}(r_S) \le \mathrm{i}(r_S \bowtie r_T) \quad \text{and} \quad \mathrm{i}(r_T) \le \mathrm{i}(r_S \bowtie r_T).$$

References

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Applications

Corollary

Let $(r_S, r_T, \alpha, \beta)$ be a matched product system of solutions. Then the following hold:

- $r_S^l = \mathrm{id}$ and $r_T^m = \mathrm{id}$, for certain $l, m \in \mathbb{N}$, if and only if $(r_S \bowtie r_T)^n = \mathrm{id}$, for a certain $n \in \mathbb{N}$;
- $r_S^l = r_S$ and $r_T^m = r_T$, for certain $l, m \in \mathbb{N}$, if and only if $(r_S \bowtie r_T)^n = r_S \bowtie r_T$ for a certain $n \in \mathbb{N}$.

This result covers the particular case of involutive solutions and the one of idempotent solutions.

Recall that a semi-brace is a set B with two operations $+$ and $\circ$ such that $(B, +)$ is a semigroup, $(B, \circ)$ is a group, and

$$a \circ (b + c) = a \circ b + a \circ (a^- + c)$$

holds for all $a, b, c \in B$ where a^- is the inverse of a in $(B, \circ)$. Under mild assumptions, it is possible to associate a solution with a semi-brace and, moreover, the semi-brace B can be written as the matched product $B = F \bowtie (G \bowtie E)$, where F is a semi-brace with additive structure a left zero-semigroup, G is a skew brace, and E is a semi-brace with additive structure a right zero-semigroup.

Corollary

Let $B = F \bowtie (G \bowtie E)$ be a semi-brace. Thus, for every $n \in \mathbb{N}$

$$r_G^n = \mathrm{id} \iff r_B^{n+1} = r_B.$$

In particular, G is a left brace if and only if $r_B^3 = r_B$.

This Corollary proves that every solution associated with a semi-brace has always index 1, improving Theorem 3.2 obtained by E. Jespers and A. Van Antwerpen.

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